Variation of Parameters for Second Order Linear Differential Equations

The solution of non-homogeneous equations is possible when a particular solution, y_p, of the equation can be found. If y_c is the general solution of the associated homogeneous equation, then we know that $y = y_c + y_p$ is the general solution of the non-homogeneous equation. The *method of variation of parameters* is a powerful general method that can be used to find such a particular solution. To be more specific (and focus on second order linear equations) we would like to find a particular solution for,

$$y'' + P(x)y' + Q(x)y = R(x) \tag{1}$$

with $R(x)$ any well-behaved function. To apply the method we need to know the general solution of the associated homogeneous equation, i.e.

$$y'' + P(x)y' + Q(x)y = 0 \tag{2}$$

Let us assume that the general solution is $y_c = C_1y_1 + C_2y_2$, with C_1 and C_2 two arbitrary constants and y_1 and y_2 two linearly independent solutions of equation (2). In the method we are described the following form is postulated for a particular solution:

$$y_p = v_1(x)y_1(x) + v_2(x)y_2(x) \tag{3}$$

i.e. the arbitrary constants become two variable functions to be found. In order to do so we simply compute first and second derivatives, The first derivative yields:

$$y_p' = v_1'y_1 + v_1y_1' + v_2'y_2 + v_2y_2' = (v_1y_1' + v_2y_2') + (v_1'y_1 + v_2'y_2)$$

We have collected terms in the above way to avoid having to handle second derivatives for v_1 and v_2. Thus, we fix the content in the second parenthesis equal to 0:

$$v_1'y_1 + v_2'y_2 = 0 \tag{4}$$

The first derivative, y_p', becomes, thus,

$$y_p' = v_1y_1' + v_2y_2' \tag{5}$$

From (5) we carry on, computing the second derivative:

$$y_p'' = v_1'y_1' + v_1y_1'' + v_2'y_2' + v_2y_2'' \tag{6}$$

Next, let us substitute (3), (5), (6) in the differential equation (1). The result is:

$$(v_1'y_1' + v_1y_1'' + v_2'y_2' + v_2y_2'') + P(x)(v_1y_1' + v_2y_2') + Q(x)(v_1y_1 + v_2y_2) = R(x)$$

$$\Downarrow$$

$$v_1[y_1'' + P(x)y_1' + Q(x)y_1] + v_2[y_2'' + P(x)y_2' + Q(x)y_2] + v_1'y_1' + v_2'y_2' = R(x)$$

Quantities inside the square brackets are null, as both y_1 and y_2 obey equation (2). The above expression becomes, thus:

$$v_1'y_1' + v_2'y_2' = R(x) \tag{7}$$

Together, equations (4) and (7) form a system for the independent quantities v_1' and v_2'. To solve this system we first compute the following determinant:

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1' \equiv W(y_1, y_2)$$

The above determinant has been indicated as $W(y_1, y_2)$ because is the *wronskian* of the two solutions y_1 and y_2. This determinant is always different from zero, as y_1 and y_2 are linearly independent. We need two more determinants to find the system's solution:

$$\begin{vmatrix} 0 & y_2 \\ R(x) & y_2' \end{vmatrix} = -y_2R(x) \quad , \quad \begin{vmatrix} y_1 & 0 \\ y_1' & R(x) \end{vmatrix} = y_1R(x)$$

The solutions are, therefore,

$$v_1' = -\frac{y_2R(x)}{W(y_1, y_2)} \quad , \quad v_2' = \frac{y_1R(x)}{W(y_1, y_2)} \tag{8}$$

These can, successively, be easily integrated to provide v_1 and v_2.

EXAMPLE 1.

Find the general solution of,

$$y'' - y' - 2y = 4x^2$$

Solution.

The associated homogeneous equation has got the following two independent solutions:

$$y_1 = \exp(2x) \quad , \quad y_2 = \exp(-x)$$

Their derivatives are:

$$y_1' = 2\exp(2x) \quad , \quad y_2' = -\exp(-x)$$

Thus, the wronskian is given by:

$$\begin{vmatrix} e^{2x} & e^{-x} \\ 2e^{2x} & -e^{-x} \end{vmatrix} = -3e^x$$

We can find v_1 and v_2 using formula (8), from which:

$$v_1' = \frac{-4x^2e^{-x}}{-3e^x} = \frac{4}{3}x^2e^{-2x} \quad , \quad v_2' = \frac{4x^2e^{2x}}{-3e^x} = -\frac{4}{3}x^2e^x$$

Eventually, after repeated integration by parts, we find:

$$v_1 = -\frac{1}{3}e^{-2x}(2x^2 + 2x + 1) \quad , \quad v_2 = -\frac{4}{3}e^x(x^2 - 2x + 2)$$

The particular solution is, therefore, given by:

$$y_p = v_1y_1 + v_2y_2 = \left[-\frac{1}{3}e^{-2x}(2x^2 + 2x + 1)\right]e^{2x} + \left[-\frac{4}{3}e^x(x^2 - 2x + 2)\right]e^{-x}$$

or, after calculations,

$$y_p = -2x^2 + 2x - 3$$

The general solution of our equation is, in the end,

$$y = C_1\exp(2x) + C_2\exp(-x) - 2x^2 + 2x - 3$$