General Properties for Second Order Linear Differential Equations

In this document I have collected a few general results concerning homogeneous linear second-order differential equations, i.e. equations of form,

$$y^{''} + P(x)y^{'} + Q(x)y = 0 \tag{1}$$

They are very interesting results that, when no technique is available for finding the analytic form of solutions, can help investigating properties of such solutions.

1 Wronskian of two independent solutions

If y_1 and y_2 are two (not necessarily independent) solutions of equation (1), then their wronskian,

$$W(y_1, y_2) \equiv \begin{vmatrix} y_1 & y_2 \\ y_1^{'} & y_2^{'} \end{vmatrix} = y_1 y_2^{'} - y_2 y_1^{'} \tag{2}$$

is either identically zero, or it has constant sign over the whole interval of validity of the two solutions.
Let us consider the first derivative of W. A straightforward derivation of formula (2) yields:

$$W^{'} = y_1^{'} y_2^{'} + y_1 y_2^{''} - y_2^{'} y_1^{'} - y_2 y_1^{''} = y_1 y_2^{''} - y_2 y_1^{''}$$

Next, given that both y_1 and y_2 are solutions of the linear second-order homogeneous equation, the following equations are valid:

$$y_1^{''} + P(x)y_1^{'} + Q(x)y_1 = 0$$
$$y_2^{''} + P(x)y_2^{'} + Q(x)y_2 = 0$$

On multiplying the first equation by y_2, the second by y_1, and subtracting the first from the second, we obtain:

$$(y_1 y_2^{''} - y_2 y_1^{''}) + P(x)(y_1 y_2^{'} - y_2 y_1^{'}) = 0$$

In the above equation we recognize the expressions for the wronskian and its first derivative; the equation, thus, can be re-written as:

$$\frac{\mathrm{d}W}{\mathrm{d}x} + P(x)W = 0$$

The general solution of this equation is:

$$W = k \exp\left(-\int P(x)\mathrm{d}x\right)$$

The exponential is always positive, then the wronskian will have the same sign of the constant k. If $k = 0$, the wronskian will be identically zero. This is what we wanted to prove. It can be further shown (but we will not do it here) that the wronskian can be zero only if the two solutions are linearly dependent. Therefore the wronskian of two linearly independent solution is never zero and has the same sign throughout the interval of validity of the two soutions.

EXAMPLE 1.
Consider the following equation:

$$y'' + y = 0$$

$\sin x$ and $\cos x$ are two linearly independent solutions of this equation. Their wronskian is:

$$W(\sin x, \cos x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1$$

i.e. it has constant sign (it is negative) throughout the whole interval of validity of $\sin x$ and $\cos x$ (which is $-\infty < x < +\infty$).

2 Zeroes of independent solutions

If y_1 and y_2 are two independent solutions of equation (1), then the zeros of this functions are distinct and occur alternatively. More specifically, y_1 vanishes exactly once between any two successive zeros of y_2.
This is quite an interesting a powerful observation on the qualitative behaviour of linear, second order equations solutions. In order to see how this is true, let us consider the expression for the wronskian, equation (2). Quantity $y_1(x)y_2'(x) - y_2(x)y_1'(x)$ has constant sign (positive or negative, it does not matter), because the two functions are linearly independent. They cannot have a common zero. Let us, in fact, suppose the opposite, and let us call this common zero with x_0. Then $y_1(x_0) = y_2(x_0) = 0$, and $w(y_1(x_0), y_2(x_0)) = y_1(x_0)y_2'(x_0) - y_2(x_0)y_1'(x_0) = 0$. But we know that the wronskian has to be different from zero, therefore x_0 cannot be a zero for both y_1 and y_2. Let us consider, then, two successive zeros of y_2, say x_1 and x_2, and let us see what happens to y_1 in the interval $x_1 < x < x_2$. At both x_1 and x_2 the wronskian is different from zero. More specifically, $w(x_1) = y_1(x_1)y_2'(x_1)$ and $w(x_2) = y_1(x_2)y_2'(x_2)$. To fix ideas, and to make the argument easier to follow, let us suppose that the wronskian is positive. We also know that $y_2'(x_1)$ will be of a different sign from $y_2'(x_2)$, because in order to go from a zero to the next, function y_2 needs to increase and decrease or vice-versa. Therefore $y_1(x_1)y_2'(x_1)$ can maintain the same sign as $y_1(x_2)y_2'(x_2)$ only if $y_1(x_1)$ has a different sign from $y_1(x_2)$. But, being y_1 a continuous function, it will necessarily have to vanish between x_1 and x_2. Furthermore y_1 will vanish exactly only once, because x_1 and x_2 are two consecutive zeros of y_2. If this were not the case, then we could apply a similar argument to y_2 to show that it would possess another zero between x_1 and x_2, thus violating the initial assumption of two successive zeros.

EXAMPLE 2.
Using again the two independent solutions $\sin x$ and $\cos x$, it is easy to check the validity of the above statement, because the sine function has exactly one zero between two successive zeros of the cosine function.

3 Reduction of standard form to normal form

The *standard form* of a linear, second order differential equation is what we have defined at (1). This can always be reduced to its *normal form*:

$$u^{''} + q(x)u = 0 \tag{3}$$

where $q(x)$ is defined through P and Q. Let us see how to operate the transformation.
We have to suppose, first, that y can be factorised as uv, with u and v two new functions:

$$\begin{aligned} y &= uv \\ y^{'} &= u^{'}v + uv^{'} \\ y^{''} &= u^{''}v + 2u^{'}v^{'} + uv^{''} \end{aligned}$$

Then we replace the above quantities in equation (1) and, successively, collect terms in $u^{''}$, $u^{'}$ and u:

$$vu^{''} + (2v^{'} + Pv)u^{'} + (v^{''} + Pv^{'} + Qv)u = 0$$

To arrive at form (3) we simply have to set $u^{'}$ coefficient to zero:

$$2v^{'} + Pv = 0 \qquad \Rightarrow \qquad v = \exp\left(-\int P(x)\mathrm{d}x\right)$$

From the above expression for v, first and second derivatives are easily computable:

$$v^{'} = -\frac{1}{2}Pv \quad , \quad v^{''} = \left(-\frac{1}{2}P^{'} + \frac{1}{4}P^2\right)v$$

The transformed equation becomes, thus,

$$vu^{''} + (-\frac{1}{2}P^{'} + \frac{1}{4}P^2 - \frac{1}{2}P^2 + Q)vu = 0$$

$$\Downarrow$$

$$u^{''} + (Q - \frac{1}{4}P^2 - \frac{1}{2}P^{'})u = 0$$

(we can divide by v because always in this case $v \neq 0$). The above equation coincides with equation (3) once $q(x)$ is defined as:

$$q(x) = Q(x) - \frac{1}{4}P^2(x) - \frac{1}{2}P^{'}(x) \tag{4}$$

<u>EXAMPLE 3</u>.
Reduce Bessel's equation,

$$x^2y^{''} + xy^{'} + (x^2 - m^2)y = 0$$

to its normal form.

<u>Solution</u>.
First the equation has to be re-written in its standard form, through a division by x^2:

$$y^{''} + \frac{1}{x}y^{'} + \left(1 - \frac{m^2}{x^2}\right)y = 0$$

Then it is simply a matter of computing $q(x)$ according to formula (4). For this purpose we compute the following quantities:

$$\begin{aligned} Q(x) &= 1 - \frac{m^2}{x^2} \\ -\frac{1}{4}P^2(x) &= -\frac{1}{4x^2} \\ -\frac{1}{2}P'(x) &= \frac{1}{2x^2} \end{aligned}$$

Therefore,

$$q(x) = 1 - \frac{m^2}{x^2} - \frac{1}{4x^2} + \frac{1}{2x^2} = 1 + \frac{1-4m^2}{4x^2}$$

and Bessel's equation in its normal form is:

$$u'' + \left(1 + \frac{1-4m^2}{4x^2}\right) u = 0$$

There is an interesting result about the normal form which we will state, but not prove. If $q(x) < 0$ and $u(x)$ is a non-trivial solution of (3), then $u(x)$ has at most one zero (that is it has not an oscillatory character). So, if one is interested in oscillations, then he/she should concentrate on equations with positive $q(x)$. Consider for example equation,

$$y'' - y = 0$$

which is already in its normal form. Here $q(x) = -1$, i.e. it is negative for all x. The general solution is $C_1 \exp(x) + C_2 \exp(-x)$. To visualise how this solution has at most one zero, let us choose any two arbitrary constants, for instance $C_1 = -C_2 = 1$; so $y(x) = \exp(x) - 1/\exp(x)$. The plot for this function is shown at Figure 1; this function clearly shows only one zero.

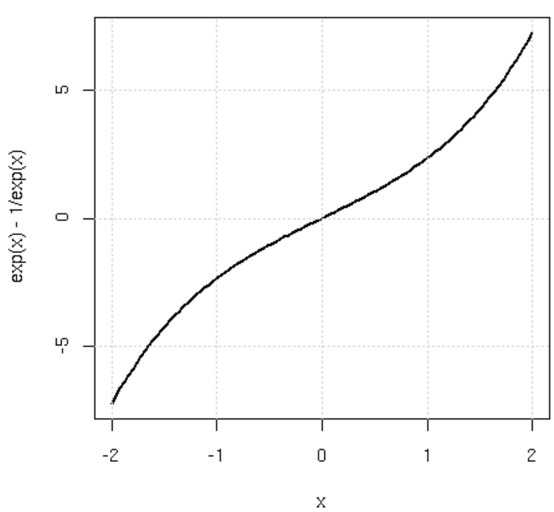

Figure 1:

4 Sturm-Liouville problems

A very important class of second order, linear differential equations goes under the name of *Sturm-Liouville problem*. More specifically, the kind of equations in this problem have the following *self-adjoint* form:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\alpha(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right] + [\lambda\beta(x) + \gamma(x)]y = 0 \tag{5}$$

where λ is a parameter that can assume several values, and is called *eigenvalue*, while functions $\alpha(x)$, $\beta(x)$ and $\gamma(x)$ are continuous on an interval $[a, b]$, and, in this interval, $\alpha(x) > 0$ and $\beta(x) > 0$. In a Sturm-Liouville problem there are boundary conditions, rather than initial conditions. In general they can be written as,

$$\begin{array}{l} C_1 y(a) + C_2 y^{'}(a) = 0 \\ D_1 y(b) + D_2 y^{'}(b) = 0 \end{array} \tag{6}$$

where, remember, a and b are boundary points for the equation interval $[a, b]$.

EXAMPLE 4.
Find a solution for the following Sturm-Liouville problem on the interval $[0, \pi]$:

$$y^{''} + \lambda y = 0 \quad , \quad \left\{ \begin{array}{l} y(0) = 0 \\ y(\pi) = 0 \end{array} \right.$$

Solution.
The given equation can be straightforwardly written in self-adjoint form as follows:

$$y^{''} + \lambda y = \frac{\mathrm{d}}{\mathrm{d}x}\left(1\frac{\mathrm{d}y}{\mathrm{d}x}\right) + (\lambda 1 + 0)y = 0$$

This is exactly form (5) with $\alpha(x) = 1 > 0$, $\beta(x) = 1 > 0$ and $\gamma(x) = 0$ a continuous function. We have, therefore, verified that the given problem is a Sturm-Liouville problem. To find solutions, let us consider, in turn, $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$.
$\lambda < 0$. If we look at the given equation in its normal form, we realize that $q(x) = \lambda < 0$. Thus we know that at most a zero is allowed for any solution. This can be the case here, because the boundary conditions force the solutio to have at least two zeros. Then λ cannot be a negative number.
$\lambda = 0$. In this case the equation is a very simple one, with general solution $y(x) = Ax + B$, i.e. a straight line. Such a curve can have at most one zero, therefore it cannot be a solution for our problem; λ cannot be zero either.
$\lambda > 0$. This is, obviously, the only interesting case. The general solution is:

$$y(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

By using the first boundary conditions, $y(0) = 0$, we get $A = 0$. Thus we are left with $y(x) = B\sin(\sqrt{\lambda}x)$. The second boundary condition becomes:

$$B\sin(\sqrt{\lambda}\pi) = 0$$

Now, b cannot be zero, otherwise the trivial solution would be the only solution to the problem. So the boundary condition transforms into:

$$\sin(\sqrt{\lambda}\pi) = 0 \quad \Rightarrow \quad \sqrt{\lambda}\pi = n\pi \ , \ n = 1, 2, 3, \ldots$$

We have found, here, a very interesting and important result for all Sturm-Liouville problems. There is usually an infinite set of solutions $y_n(x)$, called *eigenfunctions*, each one corresponding to an allowed value of the eigenvalue $\lambda = \lambda_n$. For the problem just examined, eigenvalues and eigenvectors are:

$$\lambda = \lambda_n = n^2 \quad , \quad y(x) = y_n(x) = \sin(nx) \ , \ n = 1, 2, 3, \ldots$$

Given that the starting function and the boundary conditions are linear, any linear combination of these eigenfunctions will still be solution of the Sturm-Liouville problem. There is an infinite number of eigenfunctions, thus there will exist functions, expressed as infinite summations of these eigenfunctions, that are solution to the Sturm-Liouville problem. One very important feature of the eigenfunctions coming from a Sturm-Liouville problem is that they form an orthogonal set. That is, if $y_n(x)$ and $y_m(x)$ are two eigenfunctions corresponding to eigenvalues λ_n and λ_m, then:

$$\int_a^b w(x) y_m y_n(x) \mathrm{d}x \begin{cases} = 0 & \text{if} \quad m \neq n \\ \neq 0 & \text{if} \quad m = n \end{cases} \tag{7}$$

where $w(x)$ is a so-called *weight function*, defined and continuous in interval $[a, b]$. Let us try and show that this is actually true. First of all let us re-write equation (5) for both eigenvalues:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left[\alpha(x)\frac{\mathrm{d}y_m}{\mathrm{d}x}\right] + [\lambda_m \beta(x) + \gamma(x)] y_m &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left[\alpha(x)\frac{\mathrm{d}y_n}{\mathrm{d}x}\right] + [\lambda_n \beta(x) + \gamma(x)] y_n &= 0 \end{aligned}$$

Let us now multiply the first equation by y_n, the second by y_m and subtract the second from the first:

$$y_n \frac{\mathrm{d}}{\mathrm{d}x}\left[\alpha(x)\frac{\mathrm{d}y_m}{\mathrm{d}x}\right] - y_m \frac{\mathrm{d}}{\mathrm{d}x}\left[\alpha(x)\frac{\mathrm{d}y_n}{\mathrm{d}x}\right] + (\lambda_m - \lambda_n)\beta(x) y_m y_n = 0$$

Further, let us integrate the whole equation between a and b:

$$(\lambda_m - \lambda_n)\int_a^b \beta(x) y_m(x) y_n(x) \mathrm{d}x = \int_a^b y_m(x) \frac{\mathrm{d}}{\mathrm{d}x}\left[\alpha(x)\frac{\mathrm{d}y_n}{\mathrm{d}x}\right] \mathrm{d}x - \int_a^b y_n(x) \frac{\mathrm{d}}{\mathrm{d}x}\left[\alpha(x)\frac{\mathrm{d}y_m}{\mathrm{d}x}\right] \mathrm{d}x = 0$$

Through integration by parts one obtains:

$$\int_a^b y_m(x) \frac{\mathrm{d}}{\mathrm{d}x}\left[\alpha(x)\frac{\mathrm{d}y_n}{\mathrm{d}x}\right] \mathrm{d}x = [y_m \alpha y_n']_a^b - \int_a^b y_m' \alpha y_n' \mathrm{d}x$$

and, similarly:

$$\int_a^b y_n(x) \frac{\mathrm{d}}{\mathrm{d}x}\left[\alpha(x)\frac{\mathrm{d}y_m}{\mathrm{d}x}\right] \mathrm{d}x = [y_n \alpha y_m']_a^b - \int_a^b y_n' \alpha y_m' \mathrm{d}x$$

The integrated equation so becomes:

$$(\lambda_m - \lambda_n)\int_a^b \beta(x) y_m(x) y_n(x) \mathrm{d}x = [y_m \alpha y_n']_a^b - \int_a^b y_m' \alpha y_n' \mathrm{d}x - [y_n \alpha y_m']_a^b + \int_a^b y_n' \alpha y_m' \mathrm{d}x$$

$$\Downarrow$$

$$(\lambda_m - \lambda_n)\int_a^b \beta(x) y_m(x) y_n(x) \mathrm{d}x = \alpha(b)[y_m(b) y_n'(b) - y_n(b) y_m'(b)] - \alpha(a)[y_m(a) y_n'(a) - y_n(a) y_m'(a)]$$

The quantities in square brackets are wronskians. If,

$$W_{mn}(x) \equiv \begin{vmatrix} y_m(x) & y_n(x) \\ y_m'(x) & y_n'(x) \end{vmatrix}$$

then the above equation can be re-written as follows:

$$(\lambda_m - \lambda_n)\int_a^b \beta(x)y_m(x)y_n(x)\mathrm{d}x = \alpha(b)W_{mn}(b) - \alpha(a)W_{mn}(a)$$

Now we ask: is the right hand side zero? It is certainly zero if the wronskians are zero. Let us, then, consider the boundary conditions, and re-write the first of (6) for both y_m and y_n:

$$\begin{cases} C_1 y_m(a) + C_2 y_m^{'}(a) = 0 \\ C_1 y_n(a) + C_2 y_n^{'}(a) = 0 \end{cases}$$

At least one of the two constants, C_1 and C_2, is different from zero. If we look at the above equations as a system for the two unknown C_1 and C_2, then it will give solutions other than the trivial one if the determinant,

$$\begin{vmatrix} y_m(a) & y_m^{'}(a) \\ y_n(a) & y_n^{'}(a) \end{vmatrix} \equiv W_{mn}(a)$$

is zero. Thus, $W_{mn}(a)$ must be equal to 0. Similarly, using the second of (6), we can show that $W_{mn}(b) = 0$. The integrated equation has, eventually, assumed the following form:

$$(\lambda_m - \lambda_n)\int_a^b \beta(x)y_m(x)y_n(x)\mathrm{d}x = 0$$

This is equivalent to (7) if $m \neq n$. Thus, it has been proved that the set of eigenfunctions is an orthogonal set, with weight function equal to $\beta(x)$.

EXAMPLE 5.
In EXAMPLE 4 the set of orthogonal eigenfunctions was $\{y_n(x) = \sin(nx),\ \ n = 1, 2, 3, \ldots\}$. For them the orthogonality is given by the following integral:

$$\int_0^\pi \sin(mx)\sin(nx)\mathrm{d}x = \begin{cases} 0 & \text{for} \quad m \neq n \\ \pi/2 & \text{for} \quad m = n \end{cases}$$

As $\beta(x) = 1$ in this example, we expect also the weight function to be 1. This is indeed the case in the orthogonality integral just shown.

Where does the term "self-adjoint" come from? Let us consider a generic linear, second order homogeneous equation:

$$A(x)y^{''} + B(x)y^{'} + C(x)y = 0 \tag{8}$$

Sometime it is possible to multiply the equation for a specific function, say $\mu(x)$, that change it into the following exact form:

$$\left[\mu(x)A(x)y^{'}\right]^{'} + [S(x)y]^{'} = 0 \tag{9}$$

The same function $\mu(x)$ needs to satisfy a specific differential equation; let us set up to determine such an equation. Let us, first, expand equation (9) by carrying out the derivatives:

$$\mu^{'}Ay^{'} + \mu A^{'}y^{'} + \mu Ay^{''} + S^{'}y + Sy^{'} = 0$$

$$\Downarrow$$

$$\mu Ay^{''} + (\mu^{'}A + \mu A^{'} + S)y^{'}y^{'} + S^{'}y = 0$$

A comparison of this equation with equation (8), previously multiplied by $\mu(x)$, entails the followings relations:

$$\begin{cases} \mu' A + \mu A' + S = \mu B \\ S' = \mu C \end{cases}$$

or, taking the derivative of the first equation:

$$\begin{cases} \mu'' A + \mu' A' + \mu' A' + \mu A'' + S' = \mu' B + \mu B' \\ S' = \mu C \end{cases} \Rightarrow \quad \mu'' A + 2\mu' A' + \mu A'' + \mu C = \mu' B + \mu B'$$

Thus, any chosen function $\mu(x)$, will have to obey the following equation:

$$A(x)\mu'' + [2A'(x) - B(x)]\mu' + [A''(x) - B'(x) + C(x)]\mu = 0 \tag{10}$$

This equation is called, quite sensibly, *adjoint* of equation (8). There are cases where the adjoint is exactly equivalent to the equation itself; these equation are called self-adjoints. For example, the following Legendre equation,

$$(1 - x^2)y'' - 2xy' + p(p+1)y = 0$$

has adjoint equation:

$$(1 - x^2)\mu'' + [2(-2x) - (-2x)]\mu' + [-2 - (-2) + p(p+1)]\mu = 0$$

$$\Downarrow$$

$$(1 - x^2)\mu'' - 2x\mu' + p(p+1)\mu = 0$$

This is, again, Legendre equation; thus the adjoint is equivalent to the equation itself. Therefore Legendre equation has a self-adjoint form.

Is it always possible to transform a linear, second order, homogeneous equation into a self-adjoint one? The answer is yes, as long as we multiply equation (8) by a factor:

$$\mu(x) = \frac{1}{A(x)} \exp\left[\int \frac{B(x)}{A(x)} \mathrm{d}x\right] \tag{11}$$

Things proceed as follows. Let us multiply (8) by the factor in (11):

$$\exp\left(\int \frac{B}{A}\mathrm{d}x\right) y'' + \frac{1}{A}\exp\left(\int \frac{B}{A}\mathrm{d}x\right) y' + \frac{C}{A}\exp\left(\int \frac{B}{A}\mathrm{d}x\right) y = 0$$

The first two terms can be combined into one; this way the equation assumes the form:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\exp\left(\int \frac{B}{A}\mathrm{d}x\right) y'\right] + \frac{C}{A}\exp\left(\int \frac{B}{A}\mathrm{d}x\right) y = 0$$

which is essentially the self-adjoint form (1).

<u>EXAMPLE 6</u>.
Cast Bessel's equation,

$$x^2 y'' + xy' + (x^2 - p^2)y = 0$$

into self-adjoint form.

Solution.
The whole equation will have to be multiplied by a factor given by (11). In this case it is:

$$\mu = \frac{1}{x^2} \exp\left(\int \frac{x}{x^2} \mathrm{d}x\right) = \frac{1}{x^2} \exp\left(\ln x\right) = \frac{1}{x}$$

After multiplication the equation looks like this:

$$xy^{''} + y^{'} + (x^2 - p^2)y = 0$$

The first two terms can be re-written as one. After this the equation is already in its self-adjoint form:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(xy^{'}\right) + \left(x - \frac{p^2}{x}\right) y = 0$$