Linear Differential Equations of First Order with Constant Coefficients

The general form of this type of equations is:

$$y^{'} + P(x)y = Q(x) \tag{1}$$

where $y^{'} \equiv \mathrm{d}y/\mathrm{d}x$. Equations of form,

$$A(x)y^{'} + B(x)y = C(x)$$

can easily be re-cast into form (1) simply dividing by $A(x)$.
If y_c is the general solution of the following equation, derived from (1),

$$y_c^{'} + P(x)y_c = 0, \tag{2}$$

and y_p is a particular solution of (1), then $y = y_c + y_p$ is the general solution of (1), as can be easily be verified by direct substitution of y into (1). The general solution y_c is immediately found through separation of variables:

$$\frac{\mathrm{d}y_c}{\mathrm{d}x} + P(x)y_c = 0 \quad \Rightarrow \quad \frac{\mathrm{d}y_c}{y} = -P(x)\mathrm{d}x$$

$$\Downarrow$$

$$y_c = ke^{-\int P(x)\mathrm{d}x} \tag{3}$$

with k an integration constant. To find a particular solution of (1) let us follow an intuitive argument. Any solution of (1) should be related to the solution of the associated equation (2), i.e. related to y_c. We can, then, postulate a precise form for y_p with part of it left to be determined so to satisfy the equation. More specifically, we postulate for y_p the following form:

$$y_p = u(x)e^{-\int P(x)\mathrm{d}x} \tag{4}$$

with $u(x)$ an arbitrary function to be determined. To determine $u(x)$ we simply substitute (4) into (1) and solve the new equation for u through separation of variables:

$$y_p^{'} + P(x)y_p = Q(x) \quad \Rightarrow \quad \left[u(x)e^{-\int P(x)\mathrm{d}x}\right]^{'} + P(x)\left[u(x)e^{-\int P(x)\mathrm{d}x}\right] = Q(x)$$

$$\Downarrow$$

$$u^{'}e^{-\int P(x)\mathrm{d}x} - uPe^{-\int P(x)\mathrm{d}x} + uPe^{-\int P(x)\mathrm{d}x} = Q$$

$$\Downarrow$$

$$u^{'}e^{-\int P(x)\mathrm{d}x} = Q \quad \Rightarrow \quad u^{'} = Q(x)e^{\int P(x)\mathrm{d}x}$$

The arbitrary function $u(x)$ is, thus, determined through the following integral:

$$u(x) = \int \left[Q(x) e^{\int P(x) \mathrm{d}x} \right] \mathrm{d}x \tag{5}$$

From u we can find y_p and, eventually, the general solution $y = y_c + y_p$. To summarize, the solution of a first order ordinary linear differential equations goes through the following steps:

1. Find the solution of the associated equation (2): $y_c = kE(x)$, where,

$$E(x) \equiv e^{-\int P(x) \mathrm{d}x};$$

2. Following (4) postulate a particular solution of (1) as $y_p = u(x)E(x)$;

3. Replace the postulated y_p into (1) and derive an equation for u:

$$u^{'} = Q(x) e^{\int P(x) \mathrm{d}x},$$

   which yields result (5);

4. Finally, the general solution we were looking for is $y = y_c + y_p$.

<u>EXAMPLE 1</u>.
Find the general solution of the following equation:

$$xy^{'} - 4y = x^6 e^x$$

<u>Solution</u>.
The first thing to do is to divide this equation by $A(x) = x$, so to recover form (1):

$$y^{'} - \frac{4}{x} y = x^5 e^x$$

Next we need to find $E(x) \equiv e^{-\int P(x) \mathrm{d}x}$. We have,

$$\int P(x) \mathrm{d}x = -4 \int \frac{\mathrm{d}x}{x} = -4 \ln |x| = \ln(x^{-4})$$

Therefore, $E(x) = x^4$. Now we postulate $y_p = u(x)x^4$. We, then, replace this in the original equation:

$$[ux^4]^{'} - \frac{4}{x} ux^4 = x^5 e^x \quad \Rightarrow \quad u'x^4 + 4ux^3 - 4ux^3 = x^5 e^x \quad \Rightarrow \quad u^{'} = xe^x$$

The above expression can be integrated by parts; the result is $u = e^x(x-1)$. The particular solution is, therefore, $y_p = x^4(x-1)e^x$. Finally, we have the general solution as:

$$y = y_c + y_p = kx^4 + x^5 e^x - x^4 e^x$$

<u>EXAMPLE 2</u>.
Find the solution of the following initial value problem:

$$xy^{'} + y = 2x \quad , \quad y(1) = 0$$

<u>Solution</u>.
Here $P(x) = 1/x$ and $Q(x) = 2$. Thus,

$$\int P(x) \mathrm{d}x = \int \frac{\mathrm{d}x}{x} = \ln(x)$$

(the absolute value for the logarithm is not displayed because can be considered in the integration constant). We have now found the following function E,

$$E(x) = \frac{1}{x}$$

Next, we postulate the particular solution $y_p = u(x)/x$; replacing this into the original equation yields,

$$\left[\frac{u(x)}{x}\right]' + \frac{1}{x}\frac{u(x)}{x} = 2 \qquad \Rightarrow \qquad \frac{u'}{x} - \frac{u}{x^2} + \frac{u}{x^2} = 2 \qquad \Rightarrow \qquad u' = 2x$$

The above equation obviously yields $u = x^2$, from which $y_p = x$. The general solution of the original equation is $y = x + k/x$. To find the particular solution satisfying $y(1) = 0$, we replace $x = 1$ and $y = 0$ in the general solution:

$$0 = 1 + \frac{k}{x} \qquad \Rightarrow \qquad k = -1$$

The particular solution is, therefore,

$$y = x - \frac{1}{x}$$