Laplace's Equation in Cylindrical Coordinates and Bessel's Equation (II)

1 Qualitative properties of Bessel functions of first and second kind

In the last lecture we found the expression for the general solution of Bessel's equation. More specifically, we have learnt that this solution is a linear combination of a first kind and second kind Bessel functions. We have also seen that the order of these functions could be any real number. In this lecture we are mainly interested to study the properties of Bessel functions whose order is an integer, in view of later applications.
Let us first look at the plots for Bessel functions of the first kind (see Figure 1). The following properties can be associated to these plots:

- all J_m's are oscillating functions, crossing the x-axis an infinite number of times. The zeroes

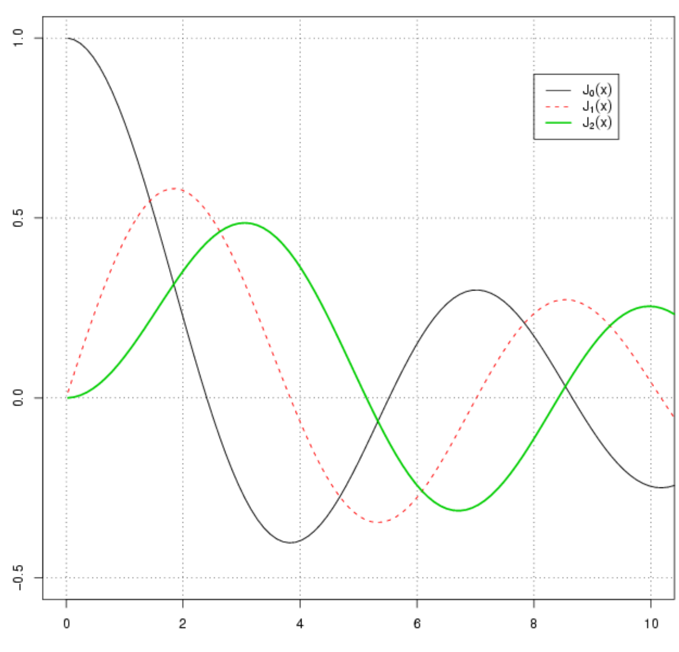

Figure 1: Plot of Bessel functions of the first kind, for orders 0, 1 and 2.

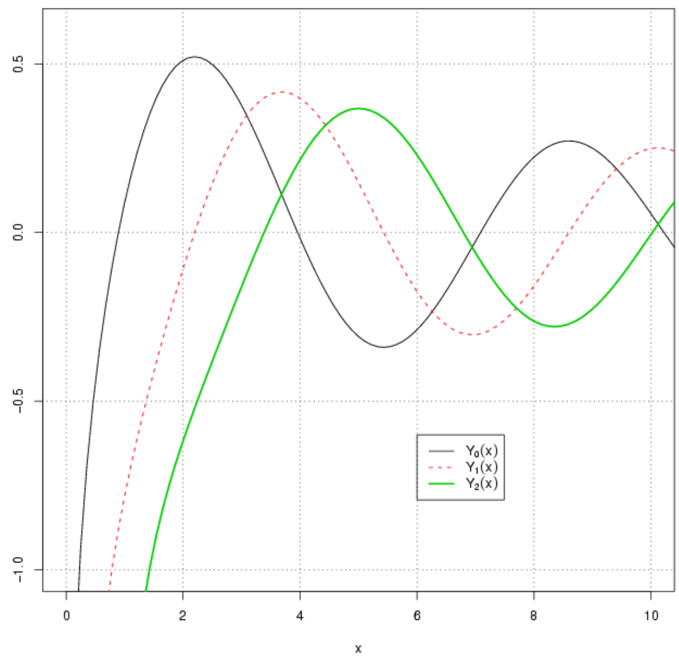

Figure 2: Plot of Bessel functions of the second kind, for orders 0, 1 and 2.

of each Bessel function of the first kind do not follow a periodic pattern, like for the sine and cosine functions;

- the amplitude of each J_m decreases at high values of x, i.e.

$$\lim_{x \to +\infty} J_m(x) = 0 \quad , \quad m = 0, 1, 2, 3, \ldots \tag{1}$$

- all J_m's tend to zero when x approaches zero. Only J_0 tends to a different value, which is 1:

$$\lim_{x \to 0} J_m(x) = 0 \; , \; m = 1, 2, 3, \ldots \quad , \quad \lim_{x \to 0} J_0(x) = 1 \tag{2}$$

- specific values for Bessel functions of the first kind are tabulated in various textbooks, or can be computed using computer algebra systems, like Maple.

Plots for the Bessel functions of the second kind are illustrated in Figure 2. For these function properties similar to Bessel functions of the first kind hold. More specifically:

- all Y_m's are oscillating functions, crossing the x-axis an infinite number of times. The zeroes of each Bessel function of the second kind do not follow a periodic pattern;

- the amplitude of each Y_m decreases at high values of x, i.e.

$$\lim_{x \to +\infty} Y_m(x) = 0 \quad , \quad m = 0, 1, 2, 3, \ldots \tag{3}$$

- all Y_m's tend to $-\infty$ when x approaches zero. In such cases we can approximate these functions with a logarithm and powers of $1/x$:

$$\lim_{x \to 0^+} Y_m(x) \sim \begin{cases} \ln(x) & m = 0 \\ 1/x^m & m = 1, 2, 3, \ldots \end{cases} \tag{4}$$

x	J_0	J_1	Y_0	Y_1
0.0	1.00000	0.00000	–	–
0.5	0.93847	0.24227	-0.44452	-1.47147
1.0	0.76520	0.44005	0.08826	-0.78121
1.5	0.51183	0.55794	0.38245	-0.41231
2.0	0.22389	0.57672	0.51038	-0.10703
2.5	-0.04838	0.49709	0.49807	0.14592
3.0	-0.26005	0.33906	0.37685	0.32467
3.5	-0.38013	0.13738	0.18902	0.41019
4.0	-0.39715	-0.06604	-0.01694	0.39793
4.5	-0.32054	-0.23106	-0.19471	0.30100
5.0	-0.17760	-0.32758	-0.30852	0.14786

Table 1: Tabulated values for some Bessel functions

- specific values for Bessel functions of the second kind are tabulated in various textbooks, or can be computed using computer algebra systems.

Some tabulated values for J_0, J_1, Y_0 and Y_1 are given in Table 1.

2 Four useful properties of Bessel functions

There are some interesting properties related to J_m and Y_m which save us the effort of looking up tabulated values too many times. These properties link in a recursive way functions of consecutive orders and their first derivatives. We can, therefore, derive values for functions of a certain order starting from values of functions of a different order.

FIRST PROPERTY.
Consider the following expression:

$$x^m J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2m}}{2^{2n+m} n! \Gamma(m+n+1)}$$

Let us calculate its first derivative:

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^m J_m(x)] = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+2m) x^{2n+2m-1}}{2^{2n+m} n! \Gamma(m+n+1)} = x^m x^{m-1} \sum_{n=0}^{\infty} \frac{(-1)^n 2(n+m) x^{2n}}{2^{2n+m} n! \Gamma(m+n+1)}$$

Now, using a Gamma function property, according to which $\Gamma(m+n+1) = (n+m)\Gamma(n+m)$, and carrying out few simplifications,

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^m J_m(x)] = x^m \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n+m-1} n! \Gamma(n+m)} \right\}$$

The quantity in brackets is $J_{m-1}(x)$. We have, thus, proved that:
Property 1

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^m J_m(x)] = x^m J_{m-1}(x) \tag{5}$$

SECOND PROPERTY.
We can prove, in an analogous fashion, that:

Property 2

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^{-m}J_m(x)] = -x^{-m}J_{m+1}(x) \tag{6}$$

In fact, we start from the expression for $x^{-m}J_m(x)$:

$$x^{-m}J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n+m} n! \Gamma(n+m+1)}$$

We then calculate the first derivative:

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^{-m}J_m(x)] = \sum_{n=0}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{2^{2n+m} n(n-1)! \Gamma(n+m+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-1}}{2^{2n+m-1}(n-1)! \Gamma(n+m+1)}$$

We know that, thanks to the properties of Gamma function, a negative factorial is an infinite quantity. Thus the first term of the summation is zero, and it will start from $n = 1$:

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^{-m}J_m(x)] = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{2^{2n+m-1}(n-1)! \Gamma(n+m+1)}$$

In order to go back to a summation starting from 0, we can replace n by $n+1$ throughout the whole expression. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^{-m}J_m(x)] = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{2^{2n+m+1} n! \Gamma(n+m+2)} = -x^{-m}\left\{ x^{m+1} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n+m+1} n! \Gamma(n+m+2)} \right\}$$

The expression in brackets is $J_{m+1}(x)$. We have, thus, derived property (6).

<u>THIRD AND FOURTH PROPERTY</u>.
Let us now expand the left-hand side of equation (5),

$$mx^{m-1}J_m + x^m J_m^{'} = x^m J_{m-1}, \tag{7}$$

and the left-hand side of equation (6

$$-mx^{-m-1}J_m + x^{-m}J_m^{'} = -x^{-m}J_{m+1}$$

Let us, also, multiply both members of this last expression by x^{2m}, to obtain

$$-mx^{m-1}J_m + x^m J_m^{'} = -x^m J_{m+1} \tag{8}$$

The third property is simply found by subtracting (8) from (7):
Property 3

$$J_{m-1}(x) + J_{m+1}(x) = \frac{2m}{x} J_m(x) \tag{9}$$

For the fourth property we have to add ((7) to (8) and divide the result by x^m:
Property 4

$$J_{m-1}(x) - J_{m+1}(x) = 2J_m^{'}(x) \tag{10}$$

Properties analogous to those just described hold for Bessel functions of the second kind, $Y_m(x)$. Let us look at how these properties can be used, in the following three examples.

EXAMPLE 1.
Find the expression for $J_{3/2}$ and $J_{-3/2}$, starting from the following known functions:

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin(x) \quad , \quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos(x) \tag{11}$$

In this particular case it is convenient to use equation (9). Replacing in it $m = 1/2$ we get:

$$J_{-1/2} + J_{3/2} = \frac{2 \cdot 1/2}{x} J_{1/2} \quad \Rightarrow \quad J_{3/2} = \frac{1}{x} J_{1/2} - J_{-1/2}$$

Finally, substituting the analytic expression for functions $J_{1/2}$ and $J_{-1/2}$:

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left[\frac{\sin(x)}{x} - \cos(x) \right]$$

We find $J_{-3/2}(x)$ using again property 3, this time with $m = -1/2$. The final result is:

$$J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}} \left[\frac{\cos(x)}{x} + \sin(x) \right]$$

EXAMPLE 2.
Compute the following integral:

$$\int_1^2 \frac{J_4(x)}{x^3} \mathrm{d}x$$

If we re-write the above integral as

$$\int_1^2 x^{-3} J_4(x) \mathrm{d}x,$$

we immediately realise that property 2 could be used advantageously. In fact:

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^{-3} J_3(x)] = -x^{-3} J_4(x)$$

and, thus

$$\int_1^2 x^{-3} J_4(x) \mathrm{d}x = -\int_1^2 \frac{\mathrm{d}}{\mathrm{d}x}[x^{-3} J_3(x)] \mathrm{d}x = -[x^{-3} J_3(x)]_1^2 = J_3(1) - \frac{J_3(2)}{8}$$

Suppose, now, we only have at our disposal Table 1. We need, then, to find J_3 as a function of J_0 and J_1. This is redaily done by applying formula (9) twice, recursively:

$$J_3(x) = \frac{4}{x} J_2(x) - J_1(x) \quad , \quad J_2(x) = \frac{2}{x} J_1(x) - J_0(x) \quad \Rightarrow \quad J_3(x) = \left(\frac{8}{x^2} - 1 \right) J_1(x) - \frac{4}{x} J_0(x)$$

We have, then, using values in Table 1,

$$\begin{aligned} J_3(1) &= 7J_1(1) - 4J_0(1) \approx 7(0.44005) - 4(0.76520) = 0.01955 \\ J_3(2) &= J_1(2) - 2J_0(2) \approx 0.57672 - 2(0.22389) = 0.12894 \end{aligned}$$

and the integral gives:

$$\int_1^2 x^{-3} J_4(x) \mathrm{d}x = J_3(1) - \frac{J_3(2)}{8} \approx 0.00343$$

EXAMPLE 3.
Find the solution of the following Bessel's equation

$$x^2R^{''} + xR^{'} + x^2R = 0$$

subject to conditions

$$R(1) = 1 \quad , \quad R^{'}(1) = -1$$

We know that the general solution of the given equation is:

$$R(x) = c_1J_0(x) + c_2Y_0(x)$$

It is left to find C_1 and c_2. From the given conditions we get:

$$\begin{cases} c_1J_0(1) + c_2Y_0(1) &= 1 \\ c_1J_0^{'}(1) + c_2Y_0^{'}(1) &= -1 \end{cases}$$

To compute $J_0^{'}$ and $Y_0^{'}$ let us use equation (6), with $m = 0$:

$$\begin{cases} J_0^{'}(x) &= -J_1(x) \\ Y_0^{'}(x) &= -Y_1(x) \end{cases} \tag{12}$$

The given conditions are then described by the following system:

$$\begin{cases} c_1J_0(1) + c_2Y_0(1) &= 1 \\ -c_1J_1(1) - c_2Y_1(1) &= -1 \end{cases}$$

which, after having replaced values from Table 1, gives $c_1 \approx 1.36576$ and $c_2 \approx -0.51074$. The solution we were looking for is, thus

$$R(x) = 1.36576J_0(x) - 0.51074Y_0(x)$$

3 Orthogonality of Bessel functions of the same order

We are already used to expressing a function as a series expansion of other functions. Taylor and Maclaurin series are common ways of expanding a function as an infinite summation of powers of x. Another typical application is the expansion in terms of sines and cosines, the Fourier series. If $f(x)$ is a periodic function of period $2L$, the associated Fourier series is defined as follows:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi}{L}x\right) + b_n\sin\left(\frac{n\pi}{L}x\right)\right] \tag{13}$$

where the coefficients a_0, a_n and b_n are calculated using the following formulas:

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x \quad , \quad b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x \quad , \quad n = 0, 1, 2, \ldots \tag{14}$$

The sines and cosines in (13) form an infinite set of functions with an interesting property:

$$\int_0^{2L}\cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\mathrm{d}x = \int_0^{2L}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\mathrm{d}x = \begin{cases} 0 & \text{if } n \neq m \\ L & \text{if } n = m \end{cases}$$

$$\int_0^{2L}\cos\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\mathrm{d}x = 0$$

i.e. the integral of the product of any two functions belonging to the infinite set is always zero, unless the two functions coincide. This property is called *orthogonality of functions* in virtue of an analogy to the scalar product of two orthogonal vectors in geometry.
In general, the set of functions $\{\varphi_0(x), \varphi_1(x), \varphi_2(x), \ldots\}$, defined in a given interval I, is said to be orthogonal if:

$$\int \varphi_n(x)\varphi_m(x)p(x)\mathrm{d}x = \begin{cases} 0 & \text{if } n \neq m \\ K & \text{if } n = m \end{cases} \tag{15}$$

where integration is performed in the interval I, $p(x)$ is a so-called *weight function*, associated to the specific set $\{\varphi_0(x), \varphi_1(x), \varphi_2(x), \ldots\}$, whose task is mainly to allow the convergence of integral (15), and K is a constant. Orthogonal functions are the favourite tool in mathematical physics when a series expansion is needed. This is due to the relative easiness of finding the coefficients associated to the expansion. In fact, let us expand an arbitrary function $f(x)$ using the general set previously introduced:

$$f(x) = \sum_{n=0}^{\infty} a_n \varphi_n(x) \tag{16}$$

Let us now multiply both members by $\varphi_m(x)p(x)$ and integrate:

$$\int f(x)\varphi_m(x)p(x)\mathrm{d}x = \sum_{n=0}^{\infty} a_n \int \varphi_n(x)\varphi_m(x)p(x)\mathrm{d}x$$

Orthogonality means that the integral on the right-hand side is zero unless $n = m$. We have, thus

$$\int f(x)\varphi_m(x)p(x)\mathrm{d}x = a_m$$

This expression gives a formula to compute all coefficients of the expansion (16):

$$a_n = \int f(x)\varphi_n(x)p(x)\mathrm{d}x \tag{17}$$

You can verify that this is true, for instance, for the Fourier series coefficients.
The reason why we have introduced the orthogonality of functions at this stage is because sets of orthogonal functions can be formed starting from Bessel functions of the first kind. We are going to prove now that, if α and β are two zeroes of any Bessel function $J_m(x)$, then the two functions $J_m(\alpha x)$ and $J_m(\beta x)$ are orthogonal. More specifically:

$$\int_0^1 xJ_m(\alpha x)J_m(\beta x)\mathrm{d}x = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ J_m'^2(\alpha)/2 & \text{if } \alpha = \beta \end{cases} \tag{18}$$

We can immediately notice that (18) is an integral of type (15), introduced to define orthogonality, with a weight function equal to x. Let us now proceed to prove result (18).
We know that a solution of the parametric Bessel's equation

$$x^2 f'' + xf' + (\lambda^2 x^2 - m^2)f = 0$$

is function $J_m(\lambda x)$. Let us, next, choose two arbitrary zeroes, α and β, of $J_m(x)$, and introduce two new functions, $u(x) = J_m(\alpha x)$ and $v(x) = J_m(\beta x)$, both solutions of the following parametric Bessel's equations:

$$x^2 u'' + xu' + (\alpha^2 x^2 - m^2)u = 0$$
$$x^2 v'' + xv' + (\beta^2 x^2 - m^2)v = 0$$

Let us multiply the first equation by v, the second by u, and subtract the second from the first. We get:

$$x^2(u^{''}v - uv^{''}) + x(u^{'}v - uv^{'}) + (\alpha^2 - \beta^2)x^2uv = 0$$

$$\Downarrow$$

$$x^2\frac{\mathrm{d}}{\mathrm{d}x}(u^{'}v - uv^{'}) + x(u^{'}v - uv^{'}) = (\beta^2 - \alpha^2)x^2uv$$

or, dividing by x and re-arranging

$$\frac{\mathrm{d}}{\mathrm{d}x}[x(u^{'}v - uv^{'})] = (\beta^2 - \alpha^2)xuv$$

Finally, integrating between 0 and 1, and using u and v definitions:

$$(\beta^2 - \alpha^2)\int_0^1 xu(x)v(x)\mathrm{d}x = [x(u^{'}v - uv^{'})]_0^1$$

$$\Downarrow$$

$$\int_0^1 xJ_m(\alpha x)J_m(\beta x)\mathrm{d}x = \frac{\alpha J_m^{'}(\alpha)J_m(\beta) - \beta J_m(\alpha)J_m^{'}(\beta)}{\beta^2 - \alpha^2} \tag{19}$$

We said that α and β are zeroes of J_m, thus $J_m(\alpha) = J_m(\beta) = 0$. If $\alpha \neq \beta$, then, the above integral is zero. Consequently, the first part of equation (18) has been proved. If $\alpha = \beta$ the fraction at (19) gives the indeterminate form 0/0. We can bypass this difficulty by using L'Hopital rule and compute the limit of the derivatives of both numerator and denominator:

$$\lim_{\beta\to\alpha} \frac{\alpha J_m^{'}(\alpha)J_m^{'}(\beta) - J_m(\alpha)J_m^{'}(\beta) - \beta J_m(\alpha)J_m^{''}(\beta)}{2\beta}$$

Given that $J_m(\alpha) = 0$, the above limit gives:

$$\frac{\alpha J_m^{'2}(\alpha)}{2\alpha} = \frac{1}{2}J_m^{'2}(\alpha)$$

Thus, also the second part of equation (18) has been proved. The result can be easily generalised to an interval $[0, a]$, with a a real, positive number. It will suffice to repeat all passages changing the integration extremes, or, more simply, to replace x with ax in integral (18). Thus, the orthogonality relation for the interval $[0, a]$, is:

$$\int_0^a xJ_m\left(\alpha\frac{x}{a}\right)J_m\left(\beta\frac{x}{a}\right)\mathrm{d}x = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ a^2J_m^{'2}(\alpha)/2 & \text{if } \alpha = \beta \end{cases} \tag{20}$$

To summarise, we have found infinite sets of functions which, starting from Bessel functions of the first kind, can be used to expand any well-behaved function between 0 and a. More specifically, if $f(x)$ is the function to be expanded, and m is the order of the Bessel functions used in the expansion process,

$$f(x) = \sum_{n=1}^{\infty} a_nJ_m\left(\lambda_n\frac{x}{a}\right) \tag{21}$$

where λ_n, $n = 1, 2, \ldots$, are the zeroes of J_m. Coefficients a_n can be calculated using orthogonality relations (20). In fact, multiplying both members of (21) by $xJ_m(\lambda_l x/a)$ and integrating between 0 and a, we obtain:

$$\int_0^a x f(x) J_m\left(\lambda_l \frac{x}{a}\right) \mathrm{d}x = \sum_{n=1}^{\infty} a_n \int_0^a x J_m\left(\lambda_n \frac{x}{a}\right) J_m\left(\lambda_l \frac{x}{a}\right) \mathrm{d}x$$

$$\Downarrow$$

$$\int_0^a x f(x) J_m\left(\lambda_l \frac{x}{a}\right) \mathrm{d}x = \frac{a_l}{2} a^2 J_m'^2(\lambda_l)$$

So we have, in general,

$$a_n = \frac{2}{a^2 J_m'^2(\lambda_n)} \int_0^a x f(x) J_m\left(\lambda_n \frac{x}{a}\right) \mathrm{d}x \tag{22}$$

Of course, we know that the only values to be tabulated are those for Bessel functions and not for their derivatives. But, using for instance properties (9) and (10), we have

$$J_m'(x) = \frac{m}{x} J_m(x) - J_{m+1}(x)$$

Now, $J_m(\lambda_n) = 0$, thus the above relations give simply $J_m'(\lambda_n) = -J_{m+1}(\lambda_n)$. Eventually:

$$a_n = \frac{2}{a^2 J_{m+1}^2(\lambda_n)} \int_0^a x f(x) J_m\left(\lambda_n \frac{x}{a}\right) \mathrm{d}x \tag{23}$$

EXAMPLE 4.
Expand, in the interval $[0, 1]$, function $1 - x^2$, using Bessel functions of the first kind of order zero.
We are looking for the coefficients of the following expression:

$$1 - x^2 = \sum_{n=1}^{\infty} a_n J_0(\lambda_n x)$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are zeroes of $J_0(x)$. using formula (23) we can compute the expression for the generic coefficient a_n:

$$a_n = \frac{2}{J_1^2(\lambda_n)} \int_0^1 x(1 - x^2) J_0(\lambda_n x) \mathrm{d}x$$

Let us integrate by parts, where $x J_0(\lambda_n x)$ is the term to be integrated and $1 - x^2$ the term to be derived. We have, first, with a change of variable, and using (5):

$$\int x J_0(\lambda_n x) \mathrm{d}x = \frac{1}{\lambda_n^2} \int t J_0(t) \mathrm{d}t = \frac{1}{\lambda_n^2} t J_1(t) = \frac{1}{\lambda_n} x J_1(\lambda_n x)$$

Integration by parts gives, then:

$$\int_0^1 x(1 - x^2) J_0(\lambda_n x) \mathrm{d}x = \frac{1}{\lambda_n} [x(1 - x^2) J_1(\lambda_n x)]_0^1 + \frac{2}{\lambda_n} \int_0^1 x^2 J_1(\lambda_1 x) \mathrm{d}x = \frac{2}{\lambda_n} \int_0^1 x^2 J_1(\lambda_1 x) \mathrm{d}x$$

Again, with a change of variable, and using (5) for a second time, we obtain

$$a_n = \frac{4}{\lambda_n J_1^2(\lambda_n)} \int_0^1 x^2 J_1(\lambda_1 x) \mathrm{d}x = \frac{4}{\lambda_n J_1^2(\lambda_n)} \frac{1}{\lambda_n^3} \int_0^1 (\lambda_n x)^2 J_1(\lambda_n x) \mathrm{d}(\lambda_n x)$$

$$= \frac{4}{\lambda_n J_1^2(\lambda_n)} \frac{1}{\lambda_n^3} [(\lambda_n x)^2 J_2(\lambda_n x)]_0^1 = \frac{4 J_2(\lambda_n)}{\lambda_n^2 J_1^2(\lambda_n)}$$

So, the final expression for $1 - x^2$ expansion is:

$$1 - x^2 = 4 \sum_{n=1}^{\infty} \frac{J_2(\lambda_n)}{\lambda_n^2 J_1^2(\lambda_n)} J_0(\lambda_n x)$$

In Figure 3 function $1 - x^2$ is plotted against the first four partial sums of the expansion. The approximation is good enough even if we consider just the first three terms.

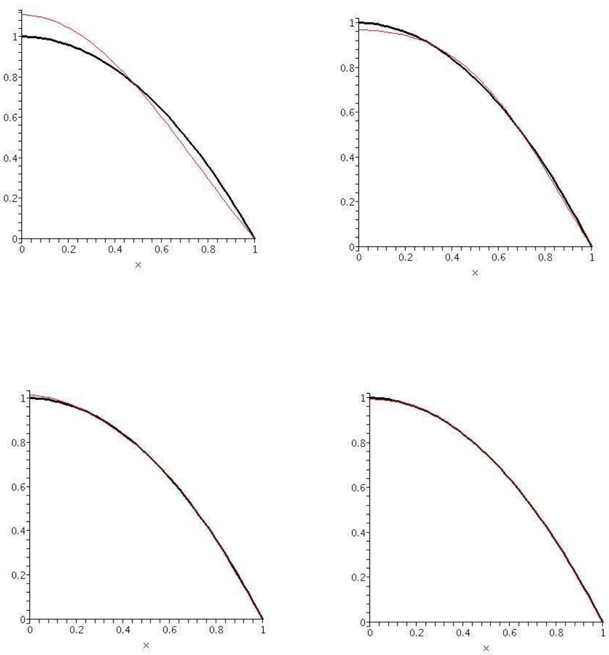

Figure 3: In these four plots, the thick curve represents function $1 - x^2$, while the thin curves represent the first four partial sums of its expansion in series of Bessel functions of the first kind, of order zero. The approximation is already good just by considering the first three terms of the expansion.

4 A boundary-value problem for Laplace's equation. Steady-state temperature in a cylinder

Laplace's equation in cylindrical coordinates has given us the opportunity of introducing and solving Bessel's equation. After having described its solutions and properties, it is now time to solve Laplace's equation in a specific case, e.g. heat conduction through a semi-infinite, solid cylinder of radius a. Let us assume that the cylinder is positioned as shown at Figure 4. The lateral side of the cylinder is constantly maintained at zero degrees centigrades, while its bottom side (the cylinder is semi-infinite) is at the uniform and constant temperature of ξ degrees centigrade. If we allow for a certain amount of time to elapse since the material at ξ degrees was first put in contact with the bottom side, then heat propagation will be described by a steady-state distribution of temperature, $u(r, \theta, z)$, obeying Laplace's equation, $\nabla^2 u = 0$. The boundary-value problem is given by the following relations:

$$\begin{cases} \nabla^2 u = 0 \\ u(a, \theta, z) = 0 \\ u(r, \theta, 0) = \xi \end{cases} \tag{24}$$

This problem, as seen in Lecture 6, is solved through separation of variables. We postulate,

$$u(r, \theta, z) = R(r)T(\theta)Z(z) \tag{25}$$

and find three different equations for r, θ and z. For the variable z we have:

$$Z(z) = a_1 e^{\ell z} + a_2 e^{-\ell z}$$

with ℓ a positive constant. Only values of z in the positive direction need to be taken care of, as the cylinder extends only in that direction. Therefore $a_1 = 0$ and the negative exponent will

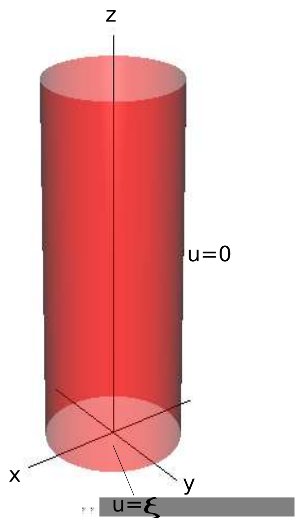

Figure 4: The temperature on the lateral surface of this semi-infinite cylinder is kept at the constant value of zero degrees centigrade. Its bottom side is kept at the constant temperature of ξ degrees centigrade. The propagation of heat, and therefore the temperature distribution $u(r, \theta, z)$ obeys Laplace's equation inside the cylinder.

make sure that the temperature decreases as we get farer and farer from the heat source at $z = 0$, a behaviour expected to occur in physical cases like this. For the θ variable, the general solution is:

$$T(\theta) = k_1 \sin(m\theta) + k_2 \cos(m\theta)$$

where m is a positive number. The temperature at θ and $\theta + 2\pi$ has to be the same. Therefore m needs to be an integer. Furthermore, given that the boundary conditions do not depend on θ, we should not notice any θ-dependence in the temperature u either, and this can be mathematically true only if this integer is zero. In such a case, in fact, the T function will become,

$$T(\theta) = k_2$$

The third equation, concerning variable r, is known to possess the general solution,

$$R(r) = c_1 J_0(\ell r) + c_2 Y_0(\ell r),$$

given that the boundary conditions have selected $m = 0$. The temperature has to be a finite quantity at $r = 0$. Thus, Y_0 cannot be included in the final solution, as it is negatively infinite at $r = 0$. To summarize, the temperature function $u(r, \theta, z)$ is given, using equation (25), by the following product,

$$u(r, \theta, z) = c_1 J_0(\ell r) k_2 a_2 e^{-\ell z} \equiv b J_0(\ell r) e^{-\ell z} \tag{26}$$

where it is clear we have merged the three constants into a single constant, b.
Let us now use the first boundary condition:

$$u(a, \theta, z) = 0 \quad \Leftrightarrow \quad b J_0(\ell a) e^{-\ell z} = 0 \quad \Rightarrow \quad J_0(\ell a) = 0$$

A Bessel function of the first kind has an infinite number of zeroes. Therefore the above equation is solved for $\ell a = \lambda_n$, $n = 1, 2, 3, \ldots$, where λ_n are the zeroes of J_0. This means that not all real values are allowed for ℓ, only a dicrete infinite number, given by,

$$\ell = \ell_n \equiv \frac{\lambda_n}{a} \quad , \quad n = 1, 2, 3, \ldots \tag{27}$$

Using equations (26) and (27) we can, thus, affirm that for this particular problem Laplace's equation has an infinite number of solutions, $u_n(r,\theta,z)$, given by:

$$u_n(r,\theta,z) = b_n J_0\left(\frac{\lambda_n}{a}r\right)e^{-\lambda_n z/a} \quad , \quad n = 1,2,3,\ldots$$

The general solution $u(r,\theta,z)$ is, consequently, given by a linear combinations of the u_n's:

$$u(r,\theta,z) = \sum_{n=1}^{\infty} b_n J_0\left(\frac{\lambda_n}{a}r\right)e^{-\lambda_n z/a} \tag{28}$$

The expansion coefficients can be computed using the last boundary condition,

$$u(r,\theta,0) = \xi \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} b_n J_0\left(\frac{\lambda_n}{a}r\right) = \xi$$

This is similar to equation (21). Thus, the expansion coefficients can be derived through the following integral:

$$b_n = \frac{2\xi}{a^2 J_1^2(\lambda_n)}\int_0^a r J_0\left(\lambda_n\frac{r}{a}\right)\mathrm{d}r$$

Through the substitution $x = \lambda_n r/a$, the above integral is transformed into,

$$b_n = \frac{2\xi}{\lambda_n^2 J_1^2(\lambda_n)}\int_0^{\lambda_n} xJ_0(x)\mathrm{d}x$$

Using the second of the four properties of Bessel functions, we can write $xJ_0(x)$ as $\mathrm{d}[xJ_1(x)]/\mathrm{d}x$. Therefore,

$$\int_0^{\lambda_n} xJ_0(x)\mathrm{d}x = [xJ_1(x)]_0^{\lambda_n} = \lambda_n J_1(\lambda_n)$$

Finally,

$$b_n = \frac{2\xi}{\lambda_n^2 J_1^2(\lambda_n)}\lambda_n J_1(\lambda_n) = \frac{2\xi}{\lambda_n J_1(\lambda_n)}$$

The complete solution of boundary-value problem (24) is:

$$u(r,\theta,z) = 2\xi\sum_{n=1}^{\infty}\frac{1}{\lambda_n J_1(\lambda_n)}J_0\left(\frac{\lambda_n}{a}r\right)e^{-\lambda_n z/a} \tag{29}$$

To develop a feeling of how this solution rightly describes a heat-propagation problem, let us limit expansion (29) to the first three terms, and fix $\xi = 200°\mathrm{C}$ and the radius $a = 10$ cm. The approximate solution will be written as,

$$\begin{aligned} u(r,\theta,z) \quad \approx \quad & 400\left[\frac{1}{\lambda_1 J_1(\lambda_1)}J_0\left(\frac{\lambda_1}{10^{-2}}r\right)e^{-\lambda_1 z/10^{-2}} + \frac{1}{\lambda_2 J_1(\lambda_2)}J_0\left(\frac{\lambda_2}{10^{-2}}r\right)e^{-\lambda_2 z/10^{-2}} + \right. \\ & \left. \frac{1}{\lambda_3 J_1(\lambda_3)}J_0\left(\frac{\lambda_3}{10^{-2}}r\right)e^{-\lambda_3 z/10^{-2}}\right] \end{aligned}$$

To compute the above expression numerically at any point (r,θ,z), we need to know the value of the first three zeroes for Bessel function J_0, and J_1 at these three values. Using tabulated values from any book (or using Maple functions *BesselJ* and *BesselJZeros*), we find:

$$\begin{array}{lll} \lambda_1 \approx 2.404826 & \lambda_2 \approx 5.520078 & \lambda_3 \approx 8.653728 \\ J_1(\lambda_1) \approx 0.519147 & J_1(\lambda_2) \approx -0.340265 & J_1(\lambda_3) \approx 0.271452 \end{array}$$

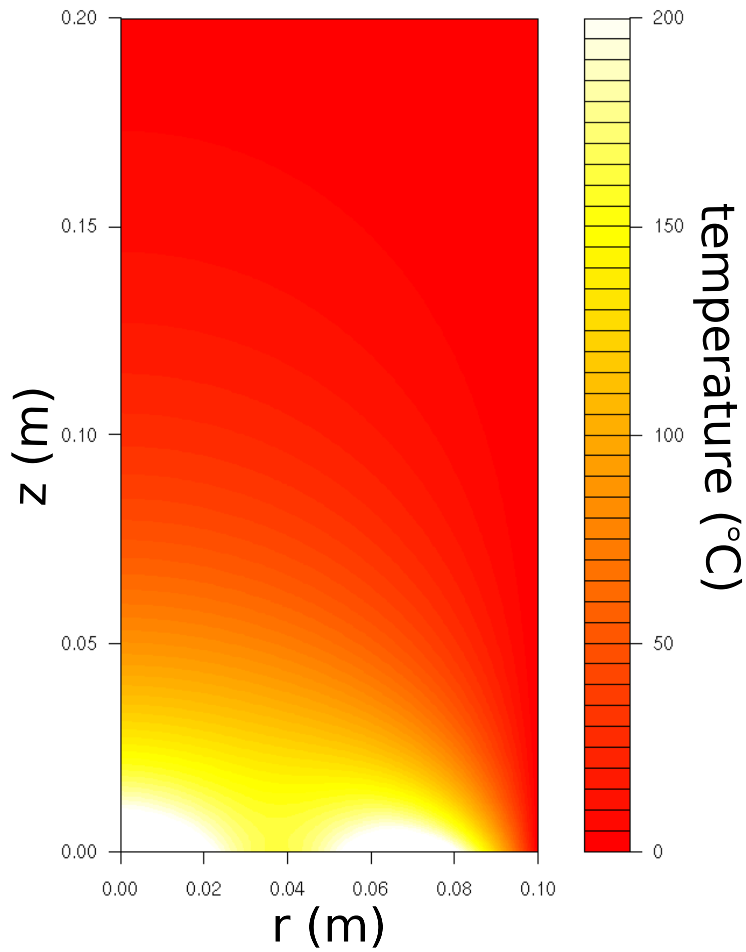

Figure 5: Temperature distribution along half a section of the cylinder. Lighter regions correspond to temperatures closer to 200 °C, darker regions to temperatures closer to 0 °C. The double bump of high temperature at the bottom of the cylinder is an artifact due to series truncation. By considering more terms in series (29), the double bump gives place to a continuous smooth high-temperature region.

So, our truncated expansion gives:

$$u(r,\theta,z) \approx 400\left[0.80099 J_0(24.04826 r)e^{-24.04826 z} - 0.53240 J_0(55.20078 r)e^{-55.20078 z} + 0.42570 J_0(86.53728 r)e^{-86.53728 z}\right]$$

From the form of this approximation, it is quite clear that each term will be damped by the exponential as the z variable increases. Therefore the temperature will be lower and lower when we proceed from the bottom to the top of the cylinder. Also, given that J_0 is 1 at $r = 0$, and goes to zero at the cylinder boundary, the temperature will be higher along the axis of the cylinder, and lower near its surface, because this is physically maintained at zero degrees centigrades. A section of the cylinder, with a filled contour-plot representing temperature distribution, is shown at Figure 5. What we have just said about the temperature distribution in the cylinder can be immediately recognised visually in the figure. The more terms will be included in series (29), the closer the temperature distribution approximate its real state.

One last comment concerns cases where the boundary conditions depend on θ. In this case the solution $u(r, \theta, z)$ will be a double summation over discrete values of sine and cosine arguments and over discrete values of ℓ. We will, in such a case, deal with an infinite set of Bessel functions and, for each function, with an infinite set of zeroes. The calculations become massive, but the conceptual level of difficulty is similar to the one of the calculation carried out here.