Laplace's Equation in Cylindrical Coordinates and Bessel's Equation (I)

1 Solution by separation of variables

Laplace's equation is a key equation in Mathematical Physics. Several phenomena involving scalar and vector fields can be described using this equation. They are mainly stationary processes, like the steady-state heat flow, described by the equation $\nabla^2 T = 0$, where $T = T(x, y, z)$ is the temperature distribution of a certain body.
The general Laplace's equation is written as:

$$\nabla^2 f = 0 \tag{1}$$

where ∇^2 is the laplacian operator. We are here mostly interested in solving Laplace's equation using cylindrical coordinates. In such a coordinate system the equation will have the following format:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

$$\Downarrow$$

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} = 0 \tag{2}$$

We will now attempt to solve equation (2) using the method of separation of variables. We start by postulating a solution having the following factorized form:

$$f(r, \theta, z) = R(r)T(\theta)Z(z) \tag{3}$$

Replacing this solution in (2) yields the intermediate result:

$$TZ\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + \frac{TZ}{r}\frac{\mathrm{d}R}{\mathrm{d}r} + \frac{RZ}{r^2}\frac{\mathrm{d}^2 T}{\mathrm{d}\theta^2} + RT\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} = 0$$

and, dividing both sides by RTZ and shifting the term containing Z on the right-hand side,

$$\frac{1}{R}\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + \frac{1}{rR}\frac{\mathrm{d}R}{\mathrm{d}r} + \frac{1}{r^2 T}\frac{\mathrm{d}^2 T}{\mathrm{d}\theta^2} = -\frac{1}{Z}\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} \tag{4}$$

In the above equation the left-hand side depends on r and θ, while the right-hand side depends on z. The only way these two members are going to be equal for all values of r, θ and z is when both of them are equal to a constant. Let us define such a constant as $-\ell^2$; it will become clear soon why we have chosen a negative number, rather than a positive one. With this choice for the constant, considering the left-hand side of equation (4), we obtain:

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} - \ell^2 Z = 0 \tag{5}$$

The general solution of this equation is:

$$Z(z) = a_1 e^{\ell z} + a_2 e^{-\ell z}$$

Such a solution, when considering the specific boundary conditions, will allow $Z(z)$ to go to zero for z going to $\pm\infty$, which makes physical sense. If we had given the constant a negative value, we would have had periodic trigonometric functions, which do not tend to zero for z going to infinity.

Once sorted the z-dependency, we need to take care of r and θ. Equation (4) reads:

$$\frac{1}{R}\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + \frac{1}{rR}\frac{\mathrm{d}R}{\mathrm{d}r} + \frac{1}{r^2 T}\frac{\mathrm{d}^2 T}{\mathrm{d}\theta^2} = -\ell^2$$

$$\Downarrow$$

$$\frac{r^2}{R}\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + \frac{r}{R}\frac{\mathrm{d}R}{\mathrm{d}r} + \ell^2 r^2 = -\frac{1}{T}\frac{\mathrm{d}^2 T}{\mathrm{d}\theta^2} \tag{6}$$

Again we are in a situation where the only way a solution can be found for the above equation is when both members are equal to a constant. This time we select a positive constant, which we call m^2. The equation for T becomes, then:

$$\frac{\mathrm{d}^2 T}{\mathrm{d}\theta^2} + m^2 T = 0 \tag{7}$$

Its general solution can be written as:

$$T(\theta) = k_1 \sin(m\theta) + k_2 \cos(m\theta)$$

This solution is well suited to describe the variation for an angular coordinate like θ. Had we chosen to set both members of equation (6) equal to a negative number, we would have ended up with exponential functions with a different value assigned to $T(\theta)$ for each 360 degrees turn, a clear non-physical solution.

Last to be examined is the r-dependency. From (6) we have:

$$\frac{r^2}{R}\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + \frac{r}{R}\frac{\mathrm{d}R}{\mathrm{d}r} + \ell^2 r^2 = m^2$$

$$\Downarrow$$

$$r^2 \frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + r\frac{\mathrm{d}R}{\mathrm{d}r} + (\ell^2 r^2 - m^2)R = 0 \tag{8}$$

Equation (8) is a well-known equation of mathematical physics called *parametric Bessel's equation*. With a simple linear transformation of variable, $x = \ell r$, equation (8) is readily changed into a *Bessel's equation*:

$$xR^{''} + xR^{'} + (x^2 - m^2)R = 0 \tag{9}$$

where $R^{''}$ and $R^{'}$ indicate first and second derivatives with respect to x. In what follows we will assume that m is a real, non-negative number. The search for a general solution of Bessel's equation will be the object of the next few chapters.

2 Frobenius method applied to Bessel's equation

The point $x = 0$ is a singular regular point for equation (9). We can therefore expand its solution R as a Frobenius series:

$$R(x) = \sum_{n=0}^{\infty} a_n x^{s+n} \tag{10}$$

The first and second derivative of this series are needed:

$$R^{'}(x) = \sum_{n=0}^{\infty} a_n (s+n) x^{s+n-1} \tag{11}$$

$$R^{''}(x) = \sum_{n=0}^{\infty} a_n (s+n)(s+n-1) x^{s+n-2} \tag{12}$$

If we replace equations (10), (11), (12) into equation (9), obtain:

$$\sum_{n=0}^{\infty} a_n(s+n)(s+n-1)x^{s+n} + \sum_{n=0}^{\infty} a_n(s+n)x^{s+n} + \sum_{n=0}^{\infty} a_n x^{s+n+2} - \sum_{n=0}^{\infty} a_n m^2 x^{s+n} = 0$$

Our next aim is to collect equal powers of x and set corresponding coefficients to zero:

$$\begin{array}{lcl}
n = 0 & \Rightarrow & a_0 s(s-1) + a_0 s - a_0 m^2 = 0 \\
n = 1 & \Rightarrow & a_1(s+1)s + a_1(s+1) - a_1 m^2 = 0 \\
n = 2 & \Rightarrow & a_2(s+2)(s+1) + a_2(s+2) + a_0 - a_2 m^2 = 0 \\
 & & \dots \\
n = k & \Rightarrow & a_k(s+k)(s+k-1) + a_k(s+k) + a_{k-2} - a_k m^2 = 0 \\
 & & \dots \\
 & & \dots
\end{array}$$

After few simplifications:

$$\left\{ \begin{array}{rcl}
a_0(s^2 - m^2) & = & 0 \\
a_1[(s+1)^2 - m^2] & = & 0 \\
a_2 & = & -a_0/[(s+2)^2 - m^2] \\
 & \dots & \\
a_k & = & -a_{k-2}/[(s+k)^2 - m^2] \\
 & \dots & \\
 & \dots &
\end{array} \right. \tag{13}$$

The term corresponding to $n = 0$ is the so-called indicial equation. Its two roots are $s = \pm m$. The Frobenius method tells us that two independent solutions, each one having form (10), can be found for equation (9) if the difference between these two roots, i.e. $m - (-m) = 2m$, is neither an integer nor zero. Let us, then, consider those cases where m is different from a multiple of 1/2. For $s = m$, from the second of equations (13), we find $a_1 = 0$. For the remaining equations we obtain:

$$a_k = -\frac{a_{k-2}}{k(k+2m)} \quad , \quad k = 2, 3, \dots \tag{14}$$

Given that $a_1 = 0$, equations (14) yields:

$$\begin{array}{rcl}
a_2 & = & -a_0/[2(2+2m)] \\
a_3 & = & -a_1/[3(3+2m)] = 0 \\
a_4 & = & -a_2/[4(4+2m)] \\
a_5 & = & -a_3/[5(5+2m)] = 0 \\
 & \dots &
\end{array}$$

Thus, all odd coefficients are zero. We can re-express even coefficients with an integer index n ranging from 1 to ∞ as,

$$a_{2n} = -a_{2n-2}/[2n(2n+2m)] = -a_{2n-2}/[2^2 n(m+n)] \quad , \quad n = 1, 2, \ldots$$

The first few coefficients will be, then:

$$a_2 = -\frac{a_0}{2^2 \cdot 1(m+1)}$$

$$a_4 = -\frac{a_2}{2^2 \cdot 2(m+2)} = -\frac{1}{2^2 \cdot 2(m+2)}\left[-\frac{a_0}{2^2 \cdot 1(m+1)}\right] = (-1)^2 \frac{a_0}{2^{2\cdot 2}(2\cdot 1)(m+2)(m+1)}$$

$$a_6 = -\frac{a_4}{2^2 \cdot 3(m+3)} = \cdots = (-1)^3 \frac{a_0}{2^{2\cdot 3}(3\cdot 2\cdot 1)(m+3)(m+2)(m+1)}$$

$$\cdots$$

$$\cdots$$

Finally, extrapolating to the n-th term:

$$a_{2n} = \frac{(-1)^n a_0}{2^{2n} n!(m+1)(m+2)\cdots(m+n)} \quad , \quad n = 1, 2, 3, \ldots \tag{15}$$

At this point we cannot give a specific value to coefficient a_0, because we are not dealing with any specific problem and have no boundary conditions which would give us the possibility to calculate it. Historically, though, it has been found convenient to standardise solutions of Bessel's equation by assigning a specific value to a_0, and express all its particular solutions as functions of the standardised ones. The choice made for a_0 is, thus:

$$a_0 = \frac{1}{2^m \Gamma(m+1)} \tag{16}$$

where $\Gamma(x)$ is the *gamma function*, defined in Appendix A. With this choice for a_0 equation (15) becomes:

$$a_{2n} = \frac{(-1)^n}{2^{2n} n!(m+1)(m+2)\cdots(m+n)} \frac{1}{2^m \Gamma(m+1)} \quad , \quad n = 1, 2, 3, \ldots$$

Now, by using recursively property (31), the above relation is transformed into:

$$a_{2n} = \frac{(-1)^n}{2^{2n+m} n!\Gamma(m+n+1)} \quad , \quad n = 1, 2, 3, \ldots \tag{17}$$

and so an independent solution of Bessel's equation is given by the following expression:

$$J_m(x) = x^m \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{m+2n} n!\Gamma(m+n+1)} \tag{18}$$

J_m is known as *Bessel function of the first kind* of order m. We will study carefully this function or, better, this set of functions, in a following lecture. At the moment we just want to find the general solution of Bessel's equation for m different from an integer or a semi-integer. Using Frobenius method we know that, with these values for m, a second solution for Bessel's function is given by J_{-m}:

$$J_{-m}(x) = \frac{1}{x^m} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{-m+2n} n!\Gamma(-m+n+1)}$$

The general solution of Bessel's equation, then, with m different from an integer or a semi-integer, is given by:

$$R(x) = c_1 J_m(x) + c_2 J_{-m}(x) \quad , \quad m \geq 0 \; , \; m \neq k\frac{1}{2} \; , \; k = 0, 1, 2, \ldots \tag{19}$$

The presence of x^m in expression (19) implies that some caution has to be used when calculating both $J_m(x)$ and $J_{-m}(x)$. First of all, $x = 0$ is ruled out from the general solution range because x^m appears at the denominator. Second, powers of negative numbers give real numbers only for integer values of the power. No real values are, in general, assigned to non-integer powers of negative numbers. For instance, $-2^{0.2}$ is a real, negative number equal to $\sqrt[5]{-2}$, while $-2^{0.7} \equiv \sqrt[10]{(-2)^7}$ is a complex number. For this reason it is safer to define solution (19) only for positive values of x, i.e. for $x > 0$.

3 Bessel functions of the first kind for m equal to semi-integers

The restriction at equation (19) can be made less strong by showing that J_m and J_{-m} are independent when m is a semi-integer. In fact we will show that, for such values of m, they can be expressed as elementary combinations of algebraic and trigonometric functions.
Consider $J_{1/2}$, first. From equation (18) we derive:

$$J_{1/2}(x) = \sqrt{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n+1/2} n! \Gamma(n+3/2)} = \sqrt{\frac{2}{x}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n+1} n! \Gamma(n+3/2)} \tag{20}$$

Let us simplify the denominator of the above expression. First of all, the gamma function can be re-written in the following way:

$$\Gamma\left(n + \frac{3}{2}\right) = \left(n + \frac{1}{2}\right) \cdot \left(n - \frac{1}{2}\right) \cdots \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right)$$

From (30) we know that $\Gamma(1/2) = \sqrt{\pi}$. Thus:

$$\Gamma\left(n + \frac{3}{2}\right) = \frac{1}{2^{n+1}} (2n+1) \cdot (2n-1) \cdots 3 \cdot 1 \cdot \sqrt{\pi} \tag{21}$$

From the denominator of (20) we also have:

$$2^{2n+1} n! = 2 \cdot 2^n \cdot 2^n \cdot n \cdot (n-1) \cdots 2 \cdot 1 = 2^{n+1} \cdot (2n) \cdot (2n-2) \cdots 4 \cdot 2 \tag{22}$$

By replacing (21) and (22) into (20) we obtain, eventually,

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sqrt{\frac{2}{\pi x}} \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right)$$

The content inside brackets, in the above expression, is easily recognizable as the MacLaurin expansion for $\sin(x)$. We have, thus,

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin(x) \tag{23}$$

In a similar way we can show that

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos(x) \tag{24}$$

From definitions (23) and (24) we can see that $J_{1/2}$ and $J_{-1/2}$ are independent functions. We can, therefore, use

$$R(x) = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)$$

as the general solution for Bessel's equation with $m = 1/2$. Indeed, all Bessel functions with m equal to a semi-integer can be expressed in terms of elementary algebraic and trigonometric functions, and, for these values of m, J_m will always be independent of J_{-m}. Sometimes, Bessel functions for semi-integer values of m are known as *spherical Bessel functions*. We can, thus, re-write the general solution (19) in the following way:

$$R(x) = c_1 J_m(x) + c_2 J_{-m}(x) \quad , \quad m \geq 0 \, , \; m \neq k \, , \; k = 0, 1, 2, \ldots \tag{25}$$

It is easy to verify (and it is left as an exercise) that the radius of convergence of series (18) is infinite. Thus, solution (25) converges for all real $x > 0$.

4 General solution of Bessel's equation for all values of m

When m is an integer, function $J_{-m}(x)$ depends linearly on J_m. Therefore, we cannot consider anymore equation (25) as the general solution of Bessel's equation. In fact, consider the expression for $J_{-m}(x)$. The gamma function becomes a factorial (see (33)) for positive $n - m + 1$, and it is infinite for negative integers. We are, then, left with:

$$J_{-m}(x) = \sum_{n=m}^{\infty} \frac{(-1)^n x^{2n-m}}{2^{2n-m} n!(n-m)!}$$

all terms in the summation from 0 to m being zero. In order to re-start the above series from $n = 0$, we need to use a new index n such that the old n is replaced in the expression by $n + m$. We get, then

$$J_{-m}(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+m} x^{2n+m}}{2^{2n+m}(n+m)!n!} = (-1)^m \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+m}}{2^{2n+m} n!(n+m)!} \equiv (-1)^m J_m(x)$$

We have thus shown that

$$J_{-m}(x) = (-1)^m J_m(x) \tag{26}$$

We cannot, therefore, use J_{-m} as a second, independent solution of Bessel's equation when applying Frobenius method. We need another solution which includes, as Frobenius method itself implies, logarithmic expressions. We will not go into a detailed explanation of this issue here, neither will derive the series expression for the second solution, containing the logarithmic term. It can be shown, more easily, that a second, linearly independent solution to Bessel's equation is given by the following *Bessel function of the second kind* of order m:

$$Y_m(x) \equiv \frac{J_m(x)\cos(m\pi) - J_{-m}(x)}{\sin(m\pi)} \tag{27}$$

If m in (27) is not an integer, it is easy to see that both Y_m and J_m are independent solutions of Bessel's equation. When m approaches an integer, however, expression (27) has the indeterminate form 0/0. Still, using L'Hopital's rule, it can be shown that

$$Y_n(x) \equiv \lim_{m \to n} Y_m(x)$$

is a function defined for all values of $x > 0$. Moreover, it is an independent solution of Bessel's equation, together with $J_n(x)$. Thus, we can re-express the general solution for all values of m, in terms of J_m and Y_m, as:

$$R(x) = c_1 J_m(x) + c_2 Y_m(x) \quad , \quad \forall m \in \mathbb{R} \quad , \quad x > 0 \tag{28}$$

More detailed description and properties of Bessel functions of first and second kind will be given in the next lecture.

EXAMPLE 1.
Find the solution of Bessel's equation

$$x^2R^{''} + xR^{'} + \left(x^2 - \frac{1}{4}\right) R = 0,$$

satisfying the following conditions:

$$R\left(\frac{\pi}{2}\right) = 1 \quad , \quad R^{'}\left(\frac{\pi}{2}\right) = -1$$

In this particular example the equation has $m = 1/2$. Using (25) we can write down its general solution as

$$R(x) = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)$$

$J_{1/2}$ and $J_{-1/2}$ are spherical Bessel functions. From expressions (23) and (24) the above general solution can be re-written as:

$$R(x) = \sqrt{\frac{2}{\pi}} \left[c_1 \frac{\sin(x)}{\sqrt{x}} + c_2 \frac{\cos(x)}{\sqrt{x}} \right]$$

Its derivative is given by

$$R^{'}(x) = \sqrt{\frac{2}{\pi}} \left[c_1 \frac{2x\cos(x) - \sin(x)}{2x\sqrt{x}} - c_2 \frac{2x\sin(x) + \cos(x)}{2x\sqrt{x}} \right]$$

Thus, the two conditions yield the following system for variables c_1 and c_2:

$$\begin{cases} \sqrt{\frac{2}{\pi}} \left[c_1 \frac{1}{\sqrt{\pi/2}} + c_2 \frac{0}{\sqrt{\pi/2}} \right] & = & 1 \\ \sqrt{\frac{2}{\pi}} \left[c_1 \frac{2(\pi/2)\cdot 0 - 1}{2(\pi/2)\sqrt{\pi/2}} - c_2 \frac{2(\pi/2)\cdot 1 - 0}{2(\pi/2)\sqrt{\pi/2}} \right] & = & -1 \end{cases}$$

From these we obtain $c_1 = \pi/2$ and $c_2 = \pi/2 - 1/2$. Thus the solution we were looking for is:

$$R(x) = \sqrt{\frac{2}{\pi}} \left[\frac{\pi}{2} \frac{\sin(x)}{\sqrt{x}} + \left(\frac{\pi}{2} - \frac{1}{2} \right) \frac{\cos(x)}{\sqrt{x}} \right]$$

EXAMPLE 2.
Find the general solution of the following parametric Bessel's equation:

$$r^2 \frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + r\frac{\mathrm{d}R}{\mathrm{d}r} + (2r^2 - 1)R = 0$$

With a change of variable, $x = \sqrt{2}r$, the above equation is transformed in the Bessel's equation

$$x^2R^{''} + xR^{'} + (x^2 - 1)R = 0$$

In this equation m is an integer. Thus, the general solution is:

$$R(x) = c_1 J_1(x) + c_2 Y_1(x)$$

The general solution for the parametric equation is derived from the above one, simply by replacing x with $\sqrt{2}r$:

$$R(r) = c_1 J_1(\sqrt{2}r) + c_2 Y_1(\sqrt{2}r)$$

Appendices

A Gamma function

The *gamma function* is defined through the following integral:

$$\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1} \mathrm{d}t \tag{29}$$

for values of ν different from 0,-1,-2,. . . , where the function is infinite. Gamma function is usually tabulated for different values of ν. An interesting and useful case, for our treatment of Bessel's functions, is the following:

$$\Gamma(1/2) = \sqrt{\pi} \tag{30}$$

The gamma function is very often used as an extension of the factorial operator to non-integer and negative numbers. In order to see that let us perform an integration by parts:

$$\Gamma(\nu+1) = \int_0^\infty e^{-t} t^{\nu} \mathrm{d}t = \left[e^{-t} t^{\nu}\right]_0^\infty + \nu \int_0^\infty e^{-t} t^{\nu-1} \mathrm{d}t = \nu\Gamma(\nu)$$

We have, thus, derived the following important property of the gamma function:

$$\Gamma(\nu+1) = \nu\Gamma(\nu) \tag{31}$$

Integral (29) for $\nu = 1$ gives:

$$\Gamma(1) = \int_0^\infty e^{-t} \mathrm{d}t = 1 \tag{32}$$

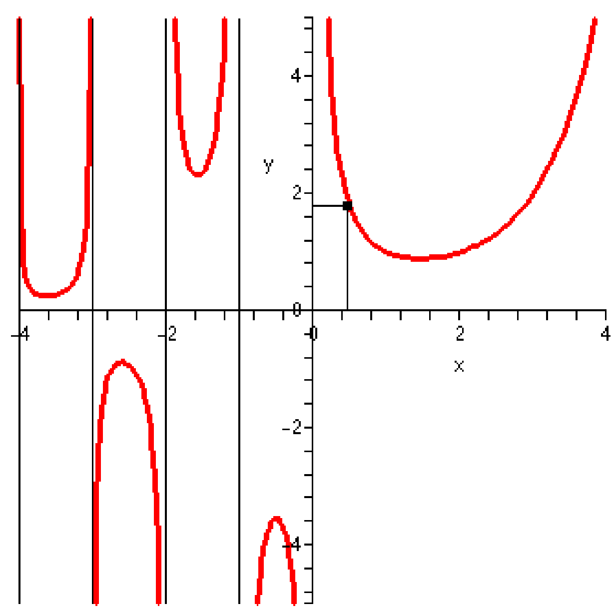

Figure 1: Plot of $\Gamma(x)$. At the point highlighted, $\Gamma(1/2) = \sqrt{\pi}$.

Starting from value (32), and using relation (31) iteratively, we obtain the following series of values:

$$
\begin{aligned}
\Gamma(1) &= 1 \\
\Gamma(2) &= 1\Gamma(1) = 1 \\
\Gamma(3) &= 2\Gamma(2) = 2 \cdot 1\Gamma(1) = 2 \cdot 1 \equiv 2! \\
\Gamma(4) &= 3\Gamma(3) = 3 \cdot 2\Gamma(2) = 3 \cdot 2 \cdot 1\Gamma(1) = 3! \\
&\ldots \\
&\ldots
\end{aligned}
$$

and so on. We have, thus, established that, for integer values, the gamma function is equivalent to the factorial function:

$$\Gamma(n+1) = n! \quad , \quad n \text{ an integer} \tag{33}$$

To conclude this Appendix on the gamma function we can show that the same relation (31) can be used to compute negative values (different from -1,-2,...) of $\Gamma(x)$. From (31) we have, in fact

$$\Gamma(\nu) = \frac{\Gamma(\nu+1)}{\nu} \tag{34}$$

So, for instance, if $\Gamma(-1.5)$ is needed, then, from (34):

$$\Gamma(-1.5) = \frac{\Gamma(-0.5)}{(-1.5)} = -\frac{1}{1.5}\frac{\Gamma(0.5)}{(-0.5)} = \frac{\sqrt{\pi}}{0.75}$$

In essence, we only need tabulated values of x between, say, 1 and 2, to determine all values of gamma.