Calculus of Variations (II)

The Euler-Lagrange equation makes it feasible to find functions which are extrema for integrals of the following form:

$$I = \int_{x_1}^{x_2} F[x, f(x), f^{'}(x)]\mathrm{d}x \tag{1}$$

In most of the applications considered one is mainly interested in finding a minimum or a maximum for expression (1). Quite often, though, the search for such an extremum is subject to a constraint. Consider, for instance, the *isoperimetric problem*, where we want to find a curve of fixed length (the constraint) which encloses the largest area. In such an example, we will have the area described by an integral like (1), and the curve length by the following condition:

$$J = \int_{x_1}^{x_2} G[x, f(x), f^{'}(x)]\mathrm{d}x = \ell \tag{2}$$

where J measures the curve length through function G, and ℓ is the curve length.
When an extremum is to be found under some constraints it is time to resort to Lagrange's method of undetermined multipliers. In the present case, where functionals appear instead of functions, the method will have to be transformed appropriately, and be adapted to expressions like (1) or (2). If we had to minimise a function like, for instance, $f(x, y, z)$, under a constraint described by $\phi(x, y, z) = 0$, we would introduce a Lagrange multiplier λ and find the minimum for the new function $f(x, y, z) + \lambda\phi(x, y, z)$. To minimise functional (1), under constraint (2), we act in a similar way by introducing a Lagrange multiplier λ and building a new functional:

$$A \equiv I + \lambda J = \int_{x_1}^{x_2} \left\{ F[x, f(x), f^{'}(x)] + \lambda G[x, f(x), f^{'}(x)] \right\} \mathrm{d}x \tag{3}$$

To find the minimum (or, more generally an extremum) for A, the following Euler-Lagrange equation has to be satisfied:

$$\frac{\partial(F + \lambda G)}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial(F + \lambda G)}{\partial f^{'}} = 0 \tag{4}$$

This last equation, together with the constraint, equation (2), is sufficient to find function f and the multiplier λ (which, by the way, does not need to be explicitly determined, as we know).

If more than one constraint is present, more Lagrange multipliers will have to be introduced and everything will be carried out exactly like in the previous case; only the number of equations to be considered will be increased. In the following section we will examine a couple of applications of variations subject to constraint.

1 The isoperimetric problem

As shortly explained in the last section, the isoperimetric problem concerns the search for a curve of fixed length which encloses the largest area. The curve is, of course, the circle. We are going

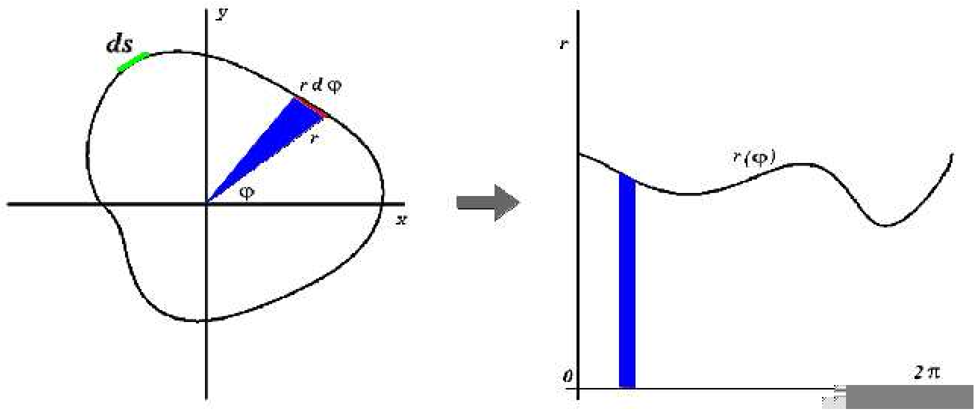

Figure 1: A closed curve in both cartesian (left) and polar coordinates. The coloured sector represents the area element, $\mathrm{d}\mathcal{A}$. This can be approximated to a right triangle whose legs are r and the short red line, $r\mathrm{d}\varphi$. The short green line represents, instead, a generic line element $\mathrm{d}s$.

to prove this statement using the Euler-Lagrange equation and Lagrange multipliers .
Given that the curve is a closed curve, proper coordinates to handle this case are the polar ones, and we will be looking for r as a function of φ, $r = r(\varphi)$. In Figure 1 a closed curve of fixed length ℓ is depicted in both cartesian and polar coordinates. The coloured element $\mathrm{d}\mathcal{A}$, swept by radius r, can be approximated by the area of a right triangle whose height is r and basis is $r\mathrm{d}\varphi$:

$$\mathrm{d}\mathcal{A} = \frac{1}{2} r^2 \mathrm{d}\varphi$$

Also, the generic line element $\mathrm{d}s$, coloured in green in Figure 1, is given by:

$$\mathrm{d}s = \sqrt{r^2(\mathrm{d}\varphi)^2 + (\mathrm{d}r)^2} = \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2}\,\mathrm{d}\varphi$$

Thus, we would like to maximise the following integral:

$$I = \int_0^{2\pi} \mathrm{d}\mathcal{A} = \frac{1}{2}\int_0^{2\pi} r^2 \mathrm{d}\varphi \tag{5}$$

subject to constraint:

$$J = \int_0^{2\pi} \mathrm{d}s = \int_0^{2\pi} \sqrt{r^2 + r'^2}\,\mathrm{d}\varphi = \ell \tag{6}$$

where $r' \equiv \mathrm{d}r/\mathrm{d}\varphi$. In this problem there is only one constraint and, thus, we only need to introduce one multiplier. The new integrand will be, then:

$$F + \lambda G = \frac{1}{2} r^2 + \lambda\sqrt{r^2 + r'^2}$$

In order to find equation (4) let us compute the two patial derivatives:

$$\frac{\partial(F + \lambda G)}{\partial r} = r + \frac{\lambda r}{\sqrt{r^2 + r'^2}} \quad , \quad \frac{\partial(F + \lambda G)}{\partial r'} = \frac{\lambda r'}{\sqrt{r^2 + r'^2}}$$

From these the Euler-Lagrange equation can be computed straightfowardly:

$$r + \frac{\lambda r}{\sqrt{r^2 + r'^2}} - \frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\frac{\lambda r'}{\sqrt{r^2 + r'^2}}\right) = 0$$

$$\Downarrow$$

$$r + \frac{\lambda r}{\sqrt{r^2 + r'^2}} - \frac{\lambda r'' \sqrt{r^2 + r'^2} - [\lambda r'^2 (r + r')/\sqrt{r^2 + r'^2}]}{r^2 + r'^2} = 0$$

This last expression can be simplified into the following one, where λ compares only on one side of the equation:

$$\frac{rr'' - 2r' - r^2}{(r^2 + r'^2)^{3/2}} = \frac{1}{\lambda} \tag{7}$$

The left-hand side of equation (7) is the well-known expression for the curvature of a curve in polar coordinates. Equation (7) tells us that such a curvature is a constant for any value of φ. The only curve with this property is the circle. We have thus shown our initial assumption was true.
In this example there has been no need to use the constraint equation a second time, thanks to the pretty form assumed by the Euler-Lagrange equation. This is, obviously, not always the case.

2 The hanging cord (catenary)

Consider a flexible, heavy cord of length 2ℓ, hanging freely from a ceiling. To simplify things let us fix a cartesian reference system on a plane perpendicular to the ceiling itself. We will assume the two ends of the cord to be at points $P_1 \equiv (-a, 0)$ and $P_2 \equiv (a, 0)$, with a a real, positive number. Let us set out to find the equation of the cord.
This problem might not immediately appear as a problem for variationl calculus. A freely hanging cord, though, will have its center of gravity in the lowest possible position. The search for the lowest y-value of the centre of gravity is, indeed, a problem that can be handled with the calculus of variations.
Supposing a uniform density ρ for the cord, its centre of gravity coordinates can be found with the following expressions:

$$\bar{x} = \int_{P_1}^{P_2} \rho x \mathrm{d}s / \int_{P_1}^{P_2} \rho \mathrm{d}s \quad , \quad \bar{y} = \int_{P_1}^{P_2} \rho y \mathrm{d}s / \int_{P_1}^{P_2} \rho \mathrm{d}s$$

with the line element ds given, as usual, by the expression $\sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}$. We are interested only in $\bar{y}$ as it needs to have the lowest allowed value. ρ is a constant, thus:

$$\bar{y} = \frac{\int_{P_1}^{P_2} \rho y \mathrm{d}s}{\int_{P_1}^{P_2} \rho \mathrm{d}s} = \frac{\int_{P_1}^{P_2} y \mathrm{d}s}{\int_{P_1}^{P_2} \mathrm{d}s}$$

The integral over the line element is equal to the cord length, 2ℓ. The above expression can consequently be re-written as:

$$\bar{y} = \frac{1}{2\ell} \int_{P_1}^{P_2} y \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}$$

and this is the expression we need to minimise.
From here we could proceed by choosing x as the independent variable and y as the dependent variable, so that $y = f(x)$. There is, though, nothing logically wrong in choosing y as the

independent variable and x as the dependent one. The cord will simply be described by a different function, $x = g(y)$. Quite often both choices are equivalent, but there are circumstances where one case is markedly better than the other. It will all be down to the form Euler-Lagrange equation assumes. In the example under examination it turns out that considering a dependency of x on y simplifies things. Let us, then, re-write the functional according to this choice:

$$I \equiv \bar{y} = \frac{1}{2\ell}\int_{P_1}^{P_2} y\sqrt{1+g'^2}\mathrm{d}y \tag{8}$$

The Lagrange multiplier can be conveniently defined as $\lambda/(2\ell)$. The new functional to minimise is, now:

$$\int_{P_1}^{P_2}\left[F(y,g,g^{'}) + (\lambda/2\ell)G(y,g,g^{'})\right]\mathrm{d}y = \frac{1}{2\ell}\int_{P_1}^{P_2}\left[y\sqrt{1+g'^2} + \lambda\sqrt{1+g'^2}\right]\mathrm{d}y$$

We can immediately notice that the integrand does not depend on g; the corresponding Euler-Lagrange equation becomes simpler, because $\partial(F+\lambda G)/\partial g = 0$. We are left with the following expression:

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{\partial(F+\lambda G)}{\partial g^{'}}\right] = 0 \;\;\Rightarrow\;\; \frac{\partial(F+\lambda G)}{\partial g^{'}} = c$$

$$\Downarrow$$

$$\frac{(y+\lambda)g^{'}}{\sqrt{1+g'^2}} = c$$

with c an integration constant. The above first-order differential equation can easily be integrated:

$$x = g(y) = \int \frac{c}{\sqrt{(y+\lambda)^2 - c^2}}\mathrm{d}y + k$$

where k is another integration constant. The integration can be easily carried out, and the general solution of the Euler-Lagrange equation is found. To determine the two integration constants we will need, at this point, to invert the function and compute $y = f(x)$, because related expression can be worked on more easily. Thus:

$$x = c\cosh^{-1}\left(\frac{y+\lambda}{c}\right) + k \qquad \Rightarrow \qquad \frac{y+\lambda}{c} = \cosh\left(\frac{x-k}{c}\right)$$

The dependency of y on x is therefore given by:

$$y = f(x) = c\cosh\left(\frac{x-k}{c}\right) - \lambda \tag{9}$$

We need, now, to determine constants c and k. By replacing the end values, $f(-a) = f(a) = 0$, the following two equations are obtained:

$$\left\{\begin{array}{rcl} c\cosh\left(\frac{-a-k}{c}\right) - \lambda & = & 0 \\ c\cosh\left(\frac{a-k}{c}\right) - \lambda & = & 0 \end{array}\right. \tag{10}$$

Comparing them yields:

$$\cosh\left(\frac{-a-k}{c}\right) = cosh\left(\frac{a-k}{c}\right)$$

If we remember that the hyperbolic cosine is defined as $\cosh(x) \equiv (e^x + e^{-x})/2$, from the above expression we derive $k = 0$. To determine c the equation for the constraint needs to be looked at. In it we can replace $f(x)$ with $c\cosh\left(\frac{x}{c}\right) - \lambda$ (remember that $k = 0$),

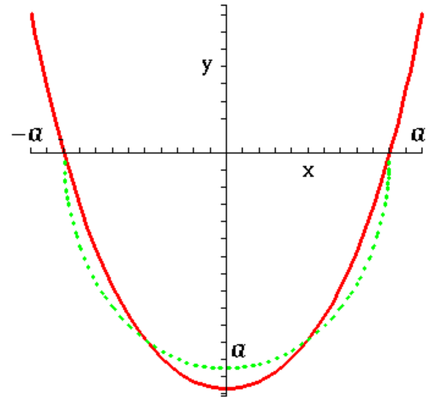

Figure 2: The catenary (full line) as compared to a semicircle (dotted line).

$$\int_{-a}^{a} \sqrt{1+f'^2(x)}\mathrm{d}x = 2\ell \quad \Rightarrow \quad \int_{-a}^{a} \sqrt{1+\sinh^2\left(\frac{x}{c}\right)}\mathrm{d}x = 0$$

$$\Downarrow$$

$$c\sinh\left(\frac{a}{c}\right) = \ell \tag{11}$$

Given the cord length, c can be computed numerically from equation (10). In order to completely define $f(x)$, we finally need to determine λ. From the second of equations (10):

$$\lambda = c\cosh\left(\frac{a}{c}\right) = c\sqrt{1+\sinh^2\left(\frac{a}{c}\right)}$$

or, using (11),

$$\lambda = c\sqrt{1+\frac{\ell^2}{c^2}} = \sqrt{c^2+\ell^2}$$

The curve we were looking for is therefore described by the following function,

$$f(x) = c\cosh\left(\frac{x}{c}\right) - \sqrt{c^2+\ell^2} \tag{12}$$

where c is a constant that can be found numerically from equation (11). This curve is known as a *catenary* and is shown in Figure 2 (against a semicircle, for visual comparison).

3 Euler-Lagrange equations: several dependent variables

When a functional is defined using only one function depending on an independent variable x, then the search for an extremum leads to a single Euler-Lagrange equation. This mostly happens when the function we are looking for describes a 2D curve. There are, obviously, many other cases where the curve (or curves) we are after are defined in more than 2 dimensions. In such cases the functionals will be defined using more than one function. The search for functional

extrema will, accordingly, be equivalent to solving several Euler-Lagrange equations.
Let us consider, then, the search for extrema of the following functional:

$$I = \int_{P_a}^{P_b} F[x, f_1(x), f_2(x), ..., f_n(x), f_1^{'}(x), f_2^{'}(x), ..., f_n^{'}(x)]\mathrm{d}x \tag{13}$$

where f_1, f_2,..., f_n are smooth functions up to the second derivative, $f_1^{'}$, $f_2^{'}$,..., $f_n^{'}$ their first derivatives, $P_a \equiv (x_a, f_1(x_a), f_2(x_a), ..., f_n(x_a))$, $P_b \equiv (x_b, f_1(x_b), f_2(x_b), ..., f_n(x_b))$, and the integration is performed along an arbitrary curve, in the n-th dimensional space, which passes through P_a and P_b. Now, we can assume that the n functions which make I an extremum are $\bar{f}_1$, $\bar{f}_2$,...,$\bar{f}_n$. As done in the one-dependent-variable case, we build a family of curves, all passing through P_a and P_b, according to the following formulae:

$$\left\{ \begin{array}{lcl} f_1(x) & \equiv & \bar{f}_1(x) + \epsilon_1\eta_1(x) \\ f_2(x) & \equiv & \bar{f}_2(x) + \epsilon_2\eta_2(x) \\ & \vdots & \\ f_n(x) & \equiv & \bar{f}_n(x) + \epsilon_n\eta_n(x) \end{array} \right. \tag{14}$$

In the above series of formulae, ϵ_1, ϵ_2,...,ϵ_n are arbitrarily small, real parameters, and $\eta_1(x)$, $\eta_2(x)$,...,$\eta_n(x)$ are arbitrary, smooth functions, obeying the following conditions:

$$\eta_1(a) = \eta_1(b) = 0 \ , \ \ \eta_2(a) = \eta_2(b) = 0 \ \ , \ldots , \ \ \eta_n(a) = \eta_n(b) = 0 \tag{15}$$

By using parametrisation (14), functional (13) becomes a function of the n variables ϵ_1, ϵ_2,...,ϵ_n. Extrema for this function will obey typical conditions:

$$\frac{\partial f_1}{\partial \epsilon_1} = 0 \ , \ \ \frac{\partial f_2}{\partial \epsilon_2} = 0 \ \ , \ldots , \ \ \frac{\partial f_n}{\partial \epsilon_n} = 0 \tag{16}$$

Let us carry out in details the full calculation for the generic component j:

$$\frac{\partial f_j}{\partial \epsilon_j} = 0$$

$$\frac{\partial}{\partial \epsilon_j} \int_{P_a}^{P_b} F[\ldots, \bar{f}_j(x) + \epsilon_j\eta_j(x), \ldots, \bar{f}_j^{'}(x) + \epsilon_j\eta_j^{'}(x), \ldots]\mathrm{d}x = 0$$

$$\Updownarrow$$

$$\int_{P_a}^{P_b} \frac{\partial}{\partial \epsilon_j} F[\ldots, \bar{f}_j(x) + \epsilon_j\eta_j(x), \ldots, \bar{f}_j^{'}(x) + \epsilon_j\eta_j^{'}(x), \ldots]\mathrm{d}x = 0$$

Using the chain-rule:

$$\int_{P_a}^{P_b} \left\{ \frac{\partial F}{\partial f_j}\eta_j + \frac{\partial F}{\partial f_j^{'}}\eta_j^{'} \right\} \mathrm{d}x = 0$$

The second integrand can be transformed using integration by parts:

$$\int_{P_a}^{P_b} \frac{\partial F}{\partial f_j^{'}}\eta_j^{'}\mathrm{d}x = \left[\frac{\partial F}{\partial f_j^{'}}\eta_j \right]_{P_a}^{P_b} - \int_{P_a}^{P_b} \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial f_j^{'}} \right) \eta\mathrm{d}x$$

and, using property (15),

$$\int_{P_a}^{P_b} \frac{\partial F}{\partial f_j^{'}}\eta_j^{'}\mathrm{d}x = - \int_{P_a}^{P_b} \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial f_j^{'}} \right) \eta\mathrm{d}x$$

The condition on component j to find an extremum is, therefore:

$$\int_{P_a}^{P_b} \left\{ \frac{\partial F}{\partial f_j} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial f_j'} \right\} \eta_j \mathrm{d}x = 0$$

But $\eta_j(x)$ is an arbitrary function, thus the previous condition is equivalent to state that the integrand is equal to zero:

$$\frac{\partial F}{\partial f_j} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial f_j'} = 0$$

This is true for all components, $j = 1, ..., n$. Thus the group of n functions of x which make functional (13) an extremum, obey the following n *Euler-Lagrange equations*:

$$\frac{\partial F}{\partial f_j} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial f_j'} = 0, \qquad j = 1, 2, ..., n \tag{17}$$

Another generalization of the search of extrema for functionals like (13) comes from making the dependent variable depending on several independent variables. It is not too difficult to find the Euler-Lagrange equation in this case, too. It will contain partial derivatives in the independent variables, instead of only one total derivative. We will not examine this further generalization here; a good description of it can be found in nearly all textbooks of mathematical physics.