Reduction of Order for Second Order Linear Differential Equations

Two functions $f_1(x)$ and $f_2(x)$, defined in a real interval I, are said to be *linearly independent* if the relation,

$$c_1 f_1(x) + c_2 f_2(x) = 0, \tag{1}$$

for all x in the interval I, can only be satisfied with $c_1 = c_2 = 0$. If, on the contrary, two non-zero constants c_1 and c_2 exist such that (1) is true for all x in I, then the functions are said to be *linearly dependent* on I. Consider two linearly dependent functions, $f_1(x)$ and $f_2(x)$, on an interval I. Relation (1) can be, then, re-written as,

$$f_2(x) = -\frac{c_1}{c_2} f_1(x) \qquad \Rightarrow \qquad \frac{f_2(x)}{f_1(x)} = \text{const}$$

because $c_1, c_2 \neq 0$. Thus if two functions are linearly dependent one of the functions is a constant multiple of the other. Conversely, if two functions are linearly independent, the ratio of the two functions cannot be a constant, but it has to vary throughout interval I:

$$\frac{f_2(x)}{f_1(x)} = u(x) \tag{2}$$

We will use the above property shortly.
Let us consider, now, the general linear second order homogeneous equation:

$$y^{''} + P(x)y^{'} + Q(x)y = 0 \tag{3}$$

and suppose one of the two solutions, y_1, is known. Given that the second solution, y_2, has to be linearly independent from the first, relation (2) tells us that $y_2/y_1 = u(x)$, i.e. we can suggest the following form for the second solution:

$$y_2 = u(x)y_1 \tag{4}$$

We want to replace y_2 in equation (3). First and second derivatives of y_2 are,

$$y_2^{'} = u^{'}y_1 + uy_1^{'} \quad , \quad y_2^{''} = uy_1^{''} + 2u^{'}y_1^{'} + u^{''}y_1$$

All this into equation (3) yields,

$$(uy_1^{''} + 2u^{'}y_1^{'} + u^{''}y_1) + P(x)(u^{'}y_1 + uy_1^{'}) + Q(x)uy_1 = 0$$

$$\Downarrow$$

$$u\underbrace{[y_1^{''} + P(x)y_1^{'} + Q(x)y_1]}_{=0} + y_1u^{''} + [2y_1^{'} + P(x)y_1]u^{'} = 0$$

We have, in short, obtained the following equation for the new function u:

$$y_1 u^{''} + [2y_1^{'} + P(x)y_1]u^{'} = 0$$

which, using a new function $w = u^{'}$, becomes the following first order equation:

$$y_1 w^{'} + [2y_1^{'} + P(x)y_1]w = 0$$

or,

$$\frac{\mathrm{d}w}{w} + \left[2\frac{y_1^{'}}{y_1} + P(x)\right]\mathrm{d}x = 0 \tag{5}$$

It is straightforward to solve equation (5) and find w; from this, through an integration, we find u and, using (4), we finally obtain the second solution y_2.

EXAMPLE.
Find the second solution of:

$$x^2 y^{''} - 3xy^{'} + 4y = 0$$

knowing that $y_1 = x^2$ is the first solution.

Solution.
The equation is immediately turned into form (3) through a division by x^2:

$$y^{''} - \frac{3}{x}y^{'} + \frac{4}{x^2}y = 0$$

so that $P(x) = -3/x$. As seen, we need to solve equation (5) for w. Given that $y_1 = x^2$, the equation turns out to be,

$$\frac{\mathrm{d}w}{w} + \left[2\frac{(x^2)^{'}}{x^2} - \frac{3}{x}\right]\mathrm{d}x = 0 \quad \Rightarrow \quad \frac{\mathrm{d}w}{w} + \left[2\frac{2x}{x^2} - \frac{3}{x}\right]\mathrm{d}x = 0$$

$$\Downarrow$$

$$\frac{\mathrm{d}w}{w} + \frac{\mathrm{d}x}{x} = 0 \quad \Rightarrow \quad w = \frac{\mathrm{const}}{x}$$

Given that $w = u^{'}$, we have, after integrating w, $u = \ln x$. Finally, the second solution we were looking for is, according to (4),

$$y_2 = x^2 \ln x$$