Linear Systems of First Order Differential Equations

1 General stuff

We will restrict our description to two functions, $x(t)$ and $y(t)$. Things can be generalized quite straightforwardly. These type of systems are generally written in the following form:

$$\left\{ \begin{array}{rcl} \frac{\mathrm{d}x}{\mathrm{d}t} & = & a_1(t)x + b_1(t)y + f_1(t) \\ \frac{\mathrm{d}y}{\mathrm{d}t} & = & a_2(t)x + b_2(t)y + f_2(t) \end{array} \right. \tag{1}$$

To be more specific, systems with at least one of f_1 or f_2 are called *non-homogeneous*. If both f_1 and f_2 are null, the systems are called *homogeneous*.
The solution of linear systems proceeds along lines similar to the solution of linear equations. We first find the general solution of the homogeneous system. If $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ are two linear independent solutions of the homogeneous system, then its general solution is given by the following formula:

$$\left\{ \begin{array}{rcl} x(t) & = & c_1x_1(t) + c_2x_2(t) \\ y(t) & = & c_1y_1(t) + c_2y_2(t) \end{array} \right. \tag{2}$$

<u>EXAMPLE 1</u>.
The two sets of functions, $(x_1, y_1) = (e^{3t}, e^{3t})$ and $(x_2, y_2) = (e^{2t}, 2e^{2t})$, are independent solutions of the following linear system:

$$\left\{ \begin{array}{rcl} \frac{\mathrm{d}x}{\mathrm{d}t} & = & 4x - y \\ \frac{\mathrm{d}y}{\mathrm{d}t} & = & 2x + y \end{array} \right.$$

This can be verified simply by substitution in the equation. For instance, focussing on the first equation:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \frac{\mathrm{d}e^{3t}}{\mathrm{d}t} = 3e^{3t}$$

and,

$$4x_1 - y_1 = 4e^{3t} - e^{3t} = 3e^{3t}$$

Therefore we have verified that,

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = 4x_1 - y_1$$

that is, we have verified that (x_1, y_1) is a solution of the first equation. Similarly one can check that (x_2, y_2) is a solution of the first equation. And, in the same way, it is easy to verify that both (x_1, y_1) and (x_2, y_2) satisfy the second equation. Furthermore the two solutions are linearly independent, because the first is not a multiple of the second. The general solution of the given system can, therefore, be expressed by:

$$\left\{ \begin{array}{rcl} x(t) & = & c_1e^{3t} + c_2e^{2t} \\ y(t) & = & c_1e^{3t} + 2c_2e^{2t} \end{array} \right.$$

For non-homogeneous systems, first we find the general solution of the associated homogeneous system, then we find a particular solution of the non-homogeneous system, (x_p, y_p), and finally write down the general solution for the non-homogeneous system as,

$$\left\{ \begin{array}{lcl} x(t) & = & c_1 x_1(t) + c_2 x_2(t) + x_p(t) \\ y(t) & = & c_1 y_1(t) + c_2 y_2(t) + y_p(t) \end{array} \right. \tag{3}$$

2 Homogeneous linear systems with constant coefficients

These have the following form:

$$\left\{ \begin{array}{lcl} \frac{\mathrm{d}x}{\mathrm{d}t} & = & a_1 x + b_1 y \\ \frac{\mathrm{d}y}{\mathrm{d}t} & = & a_2 x + b_2 y \end{array} \right. \tag{4}$$

To find a solution we can attempt to proceed like for the linear differential equations, that is we postulate a solution of form:

$$\left\{ \begin{array}{lcl} x(t) & = & Ae^{mt} \\ y(t) & = & Be^{mt} \end{array} \right.$$

(we are compelled to use the same exponential, otherwise a differentiation would never equate a linear combination of the same functions). Replacing the suggested solution into (4) yields:

$$\left\{ \begin{array}{lcl} Ame^{mt} & = & a_1 Ae^{mt} + b_1 Be^{mt} \\ Bme^{mt} & = & a_2 Ae^{mt} + b_2 Be^{mt} \end{array} \right.$$

$$\Downarrow$$

$$\left\{ \begin{array}{lcl} Am & = & a_1 A + b_1 B \\ Bm & = & a_2 A + b_2 B \end{array} \right.$$

$$\Downarrow$$

$$\left\{ \begin{array}{lcl} (a_1 - m)A + b_1 B & = & 0 \\ a_2 A + (b_2 - m)B & = & 0 \end{array} \right. \tag{5}$$

This linear algebraic system has non-trivial solutions only if the determinant,

$$\left| \begin{array}{lr} a_1 - m & b_1 \\ a_2 & b_2 - m \end{array} \right|$$

equals zero. This conditions leads to the following *associated* equation to find m:

$$m^2 - (a_1 + b_2)m + (a_1 b_2 - a_2 b_1) = 0 \tag{6}$$

According to its roots nature, this sytem will yield exponential, trigonometric or mixed algebraic-trigonometric and exponential-trigonometric solutions.

<u>EXAMPLE 2</u>.
Find the general solution of the following system:

$$\left\{ \begin{array}{lcl} \frac{\mathrm{d}x}{\mathrm{d}t} & = & x + y \\ \frac{\mathrm{d}y}{\mathrm{d}t} & = & 4x - 2y \end{array} \right.$$

<u>Solution</u>.
Using (6) we can immediately write down the associated equation:

$$m^2 + m - 6 = 0$$

whose roots are real and distinct: $m=-3$ and $m=2$. Let us write down the algebraic system (5) with $m=-3$, first:

$$\begin{cases} 4A+B &= 0 \\ 4A+B &= 0 \end{cases}$$

This system has ∞^1 solutions that can be parametrized like $A=p$, $B=-4p$; a simple non-trivial solution is $A=1$, $B=-4$. Thus, a solution of our system is $(x_1,y_1)=(e^{-3t},-4e^{-3t})$. Acting along similar lines with the other root we find $A=p$, $B=p$. Another solution of our system is, therefore, $(x_2,y_2)=(e^{2t},e^{2t})$. The general solution of the given system is, eventually,

$$\begin{cases} x(t) &= c_1e^{-3t}+c_2e^{2t} \\ y(t) &= -4c_1e^{-3t}+c_2e^{2t} \end{cases}$$

<u>EXAMPLE 3</u>.
Find the general solution of the following system:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} &= 4x-2y \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= 5x+2y \end{cases}$$

<u>Solution</u>.
The associated characteristic equation is:

$$m^2-6m+18=0 \quad \Rightarrow \quad m_1=3+3\mathrm{i}, m_2=3-3\mathrm{i}$$

Let us find A and B for $m=3+3\mathrm{i}$; these can be calculated from the first of (5):

$$(4-3-3\mathrm{i})A+2B=0 \quad \Rightarrow \quad (1-3\mathrm{i})A+2B=0$$

A solution is, for example, $A=2$, $B=1-3\mathrm{i}$. Thus, the first independent solution of the differential linear system is:

$$\begin{cases} x_1 &= 2e^{(3+3\mathrm{i})t} \\ y_1 &= (1-3\mathrm{i})e^{(3+3\mathrm{i})t} \end{cases} \quad \Rightarrow \quad \begin{cases} x_1 &= 2e^{3t}[\cos(3t)+\mathrm{i}\sin(3t)] \\ y_1 &= (1-3\mathrm{i})e^{3t}[\cos(3t)+\mathrm{i}\sin(3t)] \end{cases}$$

When $m=3-3\mathrm{i}$ we have, from the first of (5):

$$(4-3+3\mathrm{i})A+2B=0 \quad \Rightarrow \quad (1+3\mathrm{i})A+2B=0$$

which give, for example, $A=2$ and $B=1+3\mathrm{i}$. The other independent solution for our linear system is, therefore,

$$\begin{cases} x_2 &= 2e^{(3-3\mathrm{i})t} \\ y_2 &= (1+3\mathrm{i})e^{(3-3\mathrm{i})t} \end{cases} \quad \Rightarrow \quad \begin{cases} x_2 &= 2e^{3t}[\cos(3t)-\mathrm{i}\sin(3t)] \\ y_2 &= (1+3\mathrm{i})e^{3t}[\cos(3t)-\mathrm{i}\sin(3t)] \end{cases}$$

The general solution is a linear combination of these two independent solutions; we will need to fiddle a little bit with it, though, in order to transform it into its final form. So we start from:

$$\begin{cases} x(t) &= c_1x_1(t)+c_2x_2(t) &= 2c_1e^{3t}[\cos(3t)+\mathrm{i}\sin(3t)]+2c_2e^{3t}[\cos(3t)-\mathrm{i}\sin(3t)] \\ y(t) &= c_1y_1(t)+c_2y_2(t) &= c_1(1-3\mathrm{i})e^{3t}[\cos(3t)+\mathrm{i}\sin(3t)]+c_2(1+3\mathrm{i})e^{3t}[\cos(3t)-\mathrm{i}\sin(3t)] \end{cases}$$

$$\Downarrow$$

$$\begin{cases} x(t) &= 2e^{3t}[(c_1+c_2)\cos(3t)+\mathrm{i}(c_1-c_2)\sin(3t)] \\ y(t) &= e^{3t}\{[(c_1+c_2)[\cos(3t)+3\sin(3t)]+\mathrm{i}(c_1-c_2)[\sin(3t)-3\cos(3t)]\} \end{cases}$$

We can call $c_1 + c_2 \equiv H$ and $\mathrm{i}(c_1 - c_2) \equiv K$; the general solution, eventually, looks like this:

$$\begin{cases} x(t) &= 2e^{3t}[H\cos(3t) + K\sin(3t)] \\ y(t) &= e^{3t}\{[H[\cos(3t) + 3\sin(3t)] + K[\sin(3t) - 3\cos(3t)]\} \end{cases}$$

H and K can be found through the initial or boundary conditions.

EXAMPLE 4.
Find the general solution for the following system:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} &= 3x - 4y \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= x - y \end{cases}$$

Solution.
The associated characteristic equation has only one root with multiplicity two:

$$m^2 - 2m + 1 = 0 \qquad \Rightarrow \qquad m_1 = m_2 = 1$$

For the first independent solution we act as done in previous cases. Constants A and B are related by one of equations (5), for instance:

$$(3-1)A - 4B = 0 \qquad \Rightarrow \qquad A = 2B$$

An easy choice is, for instance, $A = 2$, $B = 1$. The first solution is, thus,

$$\begin{cases} x_1(t) &= 2e^t \\ y_1(t) &= e^t \end{cases}$$

To find the second solution we need to postulate it in a slightly different form the first solution. We write:

$$\begin{cases} x_2(t) &= (A_1 + A_2 t)e^t \\ y_2(t) &= (B_1 + B_2 t)e^t \end{cases}$$

To find all these coefficients we simply replace the postulated solutions in the original system of differential equations; after getting rid of the exponentials, the system looks like this:

$$\begin{cases} (A_1 + A_2 + A_2 t) &= 3(A_1 + A_2 t) - 4(B_1 + B_2 t) \\ (B_1 + B_2 + B_2 t) &= (A_1 + A_2 t) - (B_1 + B_2 t) \end{cases}$$

$$\Downarrow$$

$$\begin{cases} (-2A_2 + 4B_2)t + (-2A_1 + A_2 + 4B_1) &= 0 \\ (-A_2 + 2B_2)t + (-A_1 + 2B_1 + B_2) &= 0 \end{cases}$$

From this system, by equating equal-power terms, we obtain another system from which to derive coefficients A_1, A_2, B_1, B_2:

$$\begin{cases} -2A_2 + 4B_2 &= 0 \\ -2A_1 + A_2 + 4B_1 &= 0 \\ -A_2 + 2B_2 &= 0 \\ -A_1 + 2B_1 + B_2 &= 0 \end{cases}$$

This system has infinite valid solutions; we can simply extract one of them, for instance $A_1 = 1$, $A_2 = 2$, $B_1 = 0$, $B_2 = 1$. The second independent solution of the differential system is, thus,

$$\begin{cases} x_2(t) &= (1 + 2t)e^t \\ y_2(t) &= te^t \end{cases}$$

Finally, the general solution is:

$$\begin{cases} x(t) &= c_1 x_1(t) + c_2 x_2(t) &= 2c_1 e^t + c_2(1 + 2t)e^t \\ y(t) &= c_1 y_1(t) + c_2 y_2(t) &= c_1 e^t + c_2 te^t \end{cases}$$