Homogeneous Differential Equations of First Order

Any function of two variables, $f(x, y)$, with the following property:

$$f(tx, ty) = t^\alpha f(x, y)$$

is said to be an *homogeneous function* of degree α. If in a first order differential equations like,

$$M(x, y)\mathrm{d}x + N(x, y)\mathrm{d}y = 0$$

both $M(x, y)$ and $N(x, y)$ happen to be homogeneous functions of the same order, then the differential equation is said to be homogeneous. Consider, for instance, the following equation:

$$(x^2 + y^2)\mathrm{d}x + (x^2 - xy)\mathrm{d}y = 0$$

We have,

$$M(tx, ty) = (tx)^2 + (ty)^2 = t^2(x^2 + y^2) \equiv t^2 M(x, y)$$

and,

$$N(tx, ty) = (tx)^2 - (tx)(ty) = t^2(x^2 - xy) \equiv t^2 N(x, y)$$

Thus, both M and N are homogeneous functions of second degree so that the equation is a homogeneous equation.
Homogeneous equations are normally solved through the substitution $y = ux$ or $x = vy$, according to which one makes the resulting calculations easier. In order to show how the method works let us try to solve the equation just given. In place of y, a new dependent variable, u, is introduced as $y = ux$. For the differential we have, $\mathrm{d}y = u\mathrm{d}x + x\mathrm{d}u$. Substituting all this in the original equation we get:

$$(x^2 + u^2x^2)\mathrm{d}x + (x^2 - ux^2)(u\mathrm{d}x + x\mathrm{d}u) = 0$$

$$\Downarrow$$

$$x^2(1 + u^2)\mathrm{d}x + x^2(1 - u)u\mathrm{d}x + x^3(1 - u)\mathrm{d}u = 0$$

$$\Downarrow$$

$$[1 + u^2 + u(1 - u)]\mathrm{d}x + x(1 - u)\mathrm{d}u = 0$$

As it is immediately seen, we have obtained an equation that can be solved through separation of variables:

$$\frac{\mathrm{d}x}{x} + \frac{1 - u}{1 + u}\mathrm{d}u = 0$$

or, after an easy manipulation,

$$\frac{\mathrm{d}x}{x} + \left(-1 + \frac{2}{1 + u}\right)\mathrm{d}u = 0$$

The integration of the above equation gives:

$$-u + \ln(1+u)^2 + \ln|x| = \ln|c|,$$

where we have called the integration constant $\ln|c|$ for later easy handling. At this point we go back to our original dependent variable $y = ux$, and find:

$$-\frac{y}{x} + \ln\left(1 + \frac{y}{x}\right)^2 + \ln|x| = \ln|c|$$

or,

$$\ln\left(1 + \frac{y}{x}\right)^2 + \ln|x| - \ln|c| = \frac{y}{x}$$

and, using the logarithms properties,

$$\ln\frac{(x+y)^2}{cx} = \frac{y}{x}$$

where the absolute values have been absorbed by the arbitrariety of the integration constant c. Ultimately, the general solution is given in the following implicit form:

$$(x+y)^2 = cxe^{y/x}$$