Homogeneous Linear Differential Equations with Constant Coefficients

Among ordinary differential equations of order greater than 1, the linear homogeneous equations with constant coefficients are the most easily solvable. It is not worth to spend many words on their solution. This is a very straightforward task. First the equation is turned into an algebraic equation. Then the root of the equations are found and they become coefficients of x in the argument of an exponential. If a root has a certain multiplicity, the first independent solution is the exponential, but then a multiplication of the exponential by increasing powers of x is needed for the other independent solutions. A few examples will clarify the procedure.

EXAMPLE 1.
Find the general solution of the following equation:

$$2y^{''} - 5y^{'} - 3y = 0$$

Solution.
First we turn the equation in an algebraic one by replacing $y^{''}$ with z^2, $y^{'}$ with z and y with 1:

$$2z^2 - 5z - 3 = 0$$

This equation has roots -1/2 and 3; thus $e^{-x/2}$ and e^{3x} are two linearly independent solutions of the differential equation. The general solution is, thus,

$$y = C_1 \exp\left(-\frac{x}{2}\right) + C_2 \exp(3x)$$

EXAMPLE 2.
Find the general solution of the following equation:

$$y^{''} - 10y^{'} + 25y = 0$$

Solution.
The associated algebraic equation is:

$$z^2 - 10z + 25 = 0$$

giving a root $z = 5$ with multiplicity 2. A solution is e^{5x}. The other independent solution, which cannot be again e^{5x}, is built independently from this as xe^{5x}. Thus, the general solution is

$$y = C_1 \exp(5x) + C_2 x \exp(5x)$$

EXAMPLE 3.
Find the general solution of the following equation:

$$y^{''} + y^{'} + y = 0$$

Solution.
The associated equation has two complex conjugate roots, $-1/2 \pm \mathrm{i}\sqrt{3}/2$. The general solution could, therefore, be given straightforwardly as,

$$y = C_1 \exp\left[\left(-\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}\right) x\right] + C_2 \exp\left[\left(-\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}\right) x\right]$$

We can, however, work this expression out a little bit. First collect $\exp(-x/2)$:

$$y = \exp\left(-\frac{x}{2}\right)\left[C_1 \exp\left(\mathrm{i}\frac{\sqrt{3}}{2}x\right) + C_2 \exp\left(-\mathrm{i}\frac{\sqrt{3}}{2}x\right)\right]$$

Second, let us expand $\exp(\mathrm{i}\theta)$ according to Euler's formula, $\exp(\mathrm{i}\theta) = \cos\theta + \mathrm{i}\sin\theta$:

$$y = \exp\left(-\frac{x}{2}\right)\left[C_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + \mathrm{i}C_1 \sin\left(\frac{\sqrt{3}}{2}x\right) + C_2 \cos\left(\frac{\sqrt{3}}{2}x\right) - \mathrm{i}C_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

$$\Downarrow$$

$$y = \exp\left(-\frac{x}{2}\right)\left[(C_1 + C_2)\cos\left(\frac{\sqrt{3}}{2}x\right) + \mathrm{i}\,(C_1 - C_2)\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

Now, if we call $C_1 + C_2 = A$ and $\mathrm{i}\,(C_1 - C_2) = B$, the previous solution can be re-written as:

$$y = \exp\left(-\frac{x}{2}\right)\left[A\cos\left(\frac{\sqrt{3}}{2}x\right) + B\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

When complex conjugate roots appear, this is usually the preferred form to display results.

EXAMPLE 4.
Find the general solution of the following equation:

$$y''' + 3y'' - 4y = 0$$

Solution.
This time we have three roots for the associated algebraic equation, a simple root z=1 and a double root $z = -2$. Three linearly independent solutions are $\exp(x)$, $\exp(-2x)$ and $x\exp(-2x)$. The general solution is, thus,

$$y = C_1 \exp(x) + C_2 \exp(-2x) + C_3 x \exp(-2x)$$

EXAMPLE 5.
Find the general solution of the following equation:

$$y''' + 3y'' + 3y' + y = 0$$

Solution.
In this case $z = -1$ is the only root, with multiplicity 3, of the associated algebraic equation. Thus, three linearly independent solutions are $\exp(-x)$, $x\exp(-x)$ and $x^2\exp(-x)$. The general solution is:

$$y = C_1 \exp(-x) + C_2 x \exp(-x) + C_3 x^2 \exp(-x)$$