Exact Differential Equations

Let us consider a function of two variables, $f(x, y)$. Its differential is given by the following formula:

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y \tag{1}$$

For example, the differential of $f(x, y) = xy$ is,

$$\mathrm{d}(xy) = y\mathrm{d}x + x\mathrm{d}y$$

Some differential expressions are related to first order ordinary differential equations. For instance the following equation:

$$y^{'} = -\frac{y}{x}$$

can be re-written as,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x} \quad \Rightarrow \quad x\mathrm{d}y = -y\mathrm{d}x \quad \Rightarrow \quad y\mathrm{d}x + x\mathrm{d}y = 0 \quad \Rightarrow \quad \mathrm{d}(xy) = 0$$

i.e. as a differential equal to zero. This is, of course, not true of all first order differential equations. An equation that can be turned into a differential equal to zero is called *exact differential equation*. It is quite clear that when an equation can be turned into an exact equation, then the solution is immediately given through the implicit form:

$$f(x, y) = k \tag{2}$$

with k a constant. Thus the solution is found once $f(x, y)$ is found. There is an easy criterion to check whether an equation is exact. This criterion will be crucial in finding the general solution of the equation itself.

The differential equation needs, first, to be put in the following form:

$$M(x, y)\mathrm{d}x + N(x, y)\mathrm{d}y = 0 \tag{3}$$

This is always possible because $y^{'} \equiv \mathrm{d}y/\mathrm{d}x$. Now, comparing (3) with (1) we notice that,

$$\frac{\partial f}{\partial x} = M(x, y) \quad , \quad \frac{\partial f}{\partial y} = N(x, y) \tag{4}$$

If we take the derivative with respect to y for the first expression and the derivative with respect to x for the second expression we get:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial M}{\partial y} \quad , \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

The second order mixed partial derivatives of a function $f(x, y)$ are identical, if the function is a continuous function. Thus, for continuous functions we can write:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{5}$$

This is, actually, the condition for the reduction of a differential equation to a differential form. Consider again the differential equation previously introduced and let us re-write it according to form (3):

$$y\mathrm{d}x + x\mathrm{d}y = 0$$

Here $M(x, y) = y$ and $N(x, y) = x$. In this case condition (5) is verified as $1 = 1$, and the equation is an exact one.

Once an equation is checked to be exact, its general solution can be found in a fairly straightforward way. Integrating the first of (4) with respect to x we obtain:

$$f(x, y) = \int M(x, y)\mathrm{d}x + g(y)$$

where $g(y)$ is an arbitrary function of y (appearing here because any expression containing only the variable y is a constant with respect to an integration in $\mathrm{d}x$). In order to find $f(x, y)$ (and thus to find the implicit form of the differential equation's general solution) the function $g(y)$ needs to be found next. The derivative with respect to y can be taken of the whole expression whole expression; using also the second of (4) we get:

$$\frac{\partial f}{\partial y} = N(x, y) = \frac{\partial}{\partial y}\left[\int M(x, y)\mathrm{d}x\right] + \frac{\mathrm{d}g}{\mathrm{d}y}$$

And, rearranging this last expression,

$$\frac{\mathrm{d}g}{\mathrm{d}y} = N(x, y) - \frac{\partial}{\partial y}\left[\int M(x, y)\mathrm{d}x\right]$$

A successive integration yields the function $g(y)$. This is enough to find $f(x, y)$ and, ultimately, to find the implicit solution through equation (2).

Let us see the method on the example previously introduced. First we have to integrate with respect to x the expression:

$$\frac{\partial f}{\partial x} = y$$

This is readily done and the result is:

$$f(x, y) = xy + g(y)$$

Now we derive the last expression with respect to y and obtain:

$$\frac{\partial f}{\partial y} = N(x, y) = x + \frac{\mathrm{d}g}{\mathrm{d}y} \quad \Leftrightarrow \quad x = x + \frac{\mathrm{d}g}{\mathrm{d}y} \quad \Rightarrow \quad \frac{\mathrm{d}g}{\mathrm{d}y} = 0$$

The above expression is easily integrable, yielding $g(y) = k$, where k is a constant. Thus the general solution of the proposed equation is represented implicitely by the following relation:

$$f(x, y) = xy + k = k'$$

where k' is another integration constant. By merging k and k' into a single constant c we obtain, finally, the general solution through the following implicit relation:

$$xy = c$$

EXAMPLE 1.
Solve the following first order differential equation:

$$[e^{2y} - y\cos(xy)]\mathrm{d}x + [2xe^{2y} - x\cos(xy) + 2y]\mathrm{d}y = 0$$

Solution.
First we have to check whether the equation is exact. Here,

$$M(x,y) = e^{2y} - y\cos(xy) \quad \text{and} \quad N(x,y) = 2xe^{2y} - x\cos(xy) + 2y$$

Thus,

$$\frac{\partial M}{\partial y} = 2e^{2y} - \cos(xy) + xy\sin(xy) \quad \text{and} \quad \frac{\partial N}{\partial x} = 2e^{2y} - \cos(xy) + xy\sin(xy)$$

$$\Downarrow$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

and the equation is exact. To find the general solution we start with the following expression:

$$\frac{\partial f}{\partial x} = e^{2y} - y\cos(xy)$$

Its integration yields:

$$f(x,y) = xe^{2y} - \sin(xy) + g(y)$$

Next we use this other expression, obtained through a comparison with the derivative of the last equation:

$$N(x,y) = \frac{\partial f}{\partial y} \quad \Leftrightarrow \quad 2xe^{2y} - x\cos(xy) + 2y = 2xe^{2y} - x\cos(xy) + \frac{\mathrm{d}g}{\mathrm{d}y}$$

$$\Downarrow$$

$$\frac{\mathrm{d}g}{\mathrm{d}y} = 2y \quad \Rightarrow \quad g(y) = y^2$$

So the general solution of the equation is contained in the following implicit expression:

$$xe^{2y} - \sin(xy) + y^2 = c$$

EXAMPLE 2.
Solve the following first order differential equation:

$$(\cos x \sin x - xy^2)\mathrm{d}x + y(1 - x^2)\mathrm{d}y = 0,$$

subject to the initial condition,

$$y(0) = 2$$

Solution.
This time we have,

$$M(x,y) = \cos x \sin x - xy^2 \quad \text{and} \quad N(x,y) = y(1 - x^2)$$

Thus,

$$\frac{\partial M}{\partial y} = -2xy \quad \text{and} \quad \frac{\partial N}{\partial x} = -2xy$$

$$\Downarrow$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

and the equation is exact. To find the general solution we start with the following expression:

$$\frac{\partial f}{\partial x} = \cos x \sin x - xy^2$$

An integration with respect to x gives,

$$f(x,y) = -\frac{1}{4}\cos(2x) - \frac{1}{2}x^2y^2 + g(y)$$

Now we use the derivative with respect to y of this last expression:

$$N(x,y) = \frac{\partial f}{\partial y} \quad \Leftrightarrow \quad y(1-x^2) = -x^2y + \frac{\mathrm{d}g}{\mathrm{d}y} \quad \Rightarrow \quad \frac{\mathrm{d}g}{\mathrm{d}y} = y$$

$$\Downarrow$$

$$g(y) = \frac{1}{2}y^2$$

So, the general solution is contained in the following implicit formula:

$$\frac{1}{2}y^2(1-x^2) - \frac{1}{4}\cos(2x) = k \quad \Rightarrow \quad y^2(1-x^2) - \frac{1}{2}\cos(2x) = c$$

where we have included a factor 2 in the new constant k. The specific solution with $y(0) = 2$ can be obtained from the general solution by replacing $x = 0$ and $y = 2$:

$$2^2(1-0^2) - \frac{1}{2}\cos(2\cdot 0) = c \quad \Rightarrow \quad c = \frac{7}{2}$$

The particular solution is, therefore, analytically described by the following relation:

$$y^2(1-x^2) - \frac{1}{2}\cos(2x) = \frac{7}{2}$$