Bernoulli's Differential Equation

The general form of this type of equations is:

$$y^{'} + P(x)y = Q(x)y^n \tag{1}$$

with n any real number. The equation is easily solvable for $n = 0$ or $n = 1$. For other values of n this equation can be solved using the following substitution:

$$u = y^{1-n} \tag{2}$$

<u>EXAMPLE 1</u>.
Solve the following Bernoulli's equation:

$$xy^{'} + y = x^2y^2$$

The equation can be re-written in form (1) simply dividing by x:

$$y^{'} + \frac{1}{x}y = xy^2$$

The substitution to be used in this case is $u = y^{1-2} = 1/y$, or $y = 1/u$. Given that,

$$y^{'} = -\frac{1}{u^2}u^{'}$$

the initial equation becomes,

$$-\frac{1}{u^2}u^{'} + \frac{1}{x}\frac{1}{u} = x\frac{1}{u^2}$$

$$\Downarrow$$

$$u^{'} - \frac{1}{x}u = -x$$

We have, thus, obtained a first order linear differential equation with $P(x) = -1/x$ and $Q(x) = -x$. The general solution of this equation is:

$$u = -x^2 + cx$$

To conclude, given that $y = 1/u$, we have the following general solution for the given Bernoulli's equation:

$$y = \frac{1}{-x^2 + cx}$$